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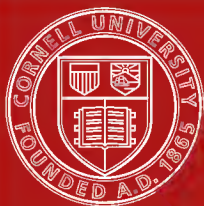
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A TEXT - BOOK
OF
APPLIED MECHANICS
AND
MECHANICAL ENGINEERING.
IN FIVE VOLUMES.

VOLUME I.—APPLIED MECHANICS.
EIGHTH EDITION, REVISED.

By PROFESSOR ANDREW JAMIESON, M.Inst.C.E., &c.

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A TEXT-BOOK
OF
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AND MECHANICAL ENGINEERING.

*Specially Arranged
For the Use of Engineers Qualifying for the Institution of Civil Engineers,
The Diplomas and Degrees of Technical Colleges and Universities,
Advanced Science Certificates of British and Colonial Boards
of Education, and Honours Certificates of the City
and Guilds of London Institute, in Mechanical
Engineering, and for Engineers Generally.*

BY
ANDREW JAMIESON, M.INST.C.E.,
FORMERLY PROFESSOR OF ENGINEERING IN THE GLASGOW AND WEST OF SCOTLAND
TECHNICAL COLLEGE; MEMBER OF THE INSTITUTION OF ELECTRICAL ENGINEERS;
FELLOW OF THE ROYAL SOCIETY, EDINBURGH; AUTHOR OF TEXT-BOOKS
ON STEAM AND STEAM ENGINES, APPLIED MECHANICS AND
MECHANICAL ENGINEERING, MAGNETISM AND
ELECTRICITY, ELECTRICAL RULES
AND TABLES, ETC.

VOLUME I.—APPLIED MECHANICS.

EIGHTH EDITION, REVISED.

**With Numerous Diagrams, Special Plates, and
Examination Questions.**



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P R E F A C E

TO THE EIGHTH EDITION OF VOL. I. APPLIED MECHANICS.

THE call for a new Edition within a year of the issue of the previous one is most gratifying. It shows that my efforts to subdivide this large subject into five separate volumes have been duly appreciated by Engineers, Teachers, and Students.

Moreover, the Reviews were most favourable and suggestive. This was especially the case in the *Engineering Supplement of the London Times* for June 15th, 1910; and in *The Electrical Review of London* for August 19th, 1910.

The latest General Instructions by The Board of Education, by The City and Guilds of London, and by The Institution of Civil Engineers, together with their respective syllabuses, have been printed in Appendix A.

Many answers have been found to previously unanswered Ordinary and C.E. Questions. These have been duly arranged and tabulated by the numbers of the various Lectures in Appendix B under two main headings.

1. The Board of Education and The City and Guilds of London.
2. The Institution of Civil Engineers for their A.M.I.C.E. Examinations.

In connection with this laborious process, I have to thank Mr. Ernest Headly Sprague, Assoc.M.Inst.C.E., of University

College, London. He kindly sent me some of his answers to compare with those obtained by me through correcting the "Correspondence Tuition Papers" of my Civil Engineering Students.

Owing to the necessity for all Engineers becoming correctly acquainted with Electrical Engineering terms and units of measurements, I have now added to Appendix D the definitions and symbol letters for the Fundamental Units of the C.G.S. System, Derived Mechanical Units and Practical Electrical Units.

The whole book has been most carefully revised, and all the latest examination papers obtainable have been added, in order to bring it up-to-date.

I have again to thank my chief assistant, Mr. John Ramsay, A.M.Inst.C.E., for his help with this new Edition, and the several Reviewers as well as others for their favourable criticisms and reception of the previous one.

ANDREW JAMIESON.

16 ROSSLYN TERRACE, KELVINSIDE,
GLASGOW, *October*, 1910.

INSTRUCTIONS FOR ANSWERING HOME EXERCISES.

(As used in my *Engineering and Electrical Science Correspondence System*.)

1. Use ordinary foolscap paper, and write on the left side *only*, leaving the facing page blank for my corrections and remarks.

2. Put the date of the Exercises at the left-hand top corner; your Name and Address in full, the name of the Subject or Section, as well as number of Lecture or Exercise, in the centre of the first page. The number of each page should be put in the right-hand top corner.

3. Leave a margin $1\frac{1}{2}$ inches wide on the left-hand side of each page, and in this margin place the *number* of the question and *nothing more*. Also, leave a clear space of *at least 2 inches* deep between your answers.

4. Be sure you understand *exactly* what the question requires you to answer, then give *all* it requires, but *no more*. If unable to fully answer any question, write down your own best attempt and state your difficulties.

5. Make your answers concise, clear, and exact, and always accompany them, if possible, by an *illustrative sketch*. Try to give (1) Side View, (2) Plan, (3) End View. Where asked, or advisable, give Sections, or Half Outside Views and Half-Sections for (1), (2), and (3).

6. Make all sketches large, open, and in the centre of the page. Do not crowd any writing about them. Simply print sizes and index letters (or names of parts), with a bold Sub-heading of what each figure or set of figures represent.

7. The character of the sketches will be carefully considered in awarding marks to the several answers. Neat sketches and "*index letters*," having the first letter of the name of the part, will always receive more marks than a bare written description.

8. All students are strongly recommended to *first* thoroughly study the Lecture, *second* to work out each answer in scroll, and *third* to compare it with the question and the text-book in order to see that each item has been answered. The final copy should be done without *any aid or reference* to the scroll copy or to *any text-book*. This acts as the best preparation for sitting at Examinations.

9. Reasonable and easily intelligible contractions (*e.g.*, mathematical, mechanical or electrical, and chemical symbols) are permitted.

10. Each corrected answer which has the symbols R. W. marked thereupon must be carefully re-worked as *one* of the set of answers for the following week.

ANDREW JAMIESON.

16 ROSSLYN TERRACE, KELVINSIDE, GLASGOW.

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MECHANICAL ENGINEERING SYMBOLS, ABBREVIATIONS, AND INDEX LETTERS

USED IN VOLUMES I. TO V.

OF PROFESSOR JAMIESON'S "APPLIED MECHANICS."

Prefatory Note.—It is very tantalising, as well as a great inconvenience to Students and Engineers, to find so many different symbol letters and terms used for denoting one and the same thing by various writers on mechanics. It is a pity, that British Civil and Mechanical Engineers have not as yet *standardised* their symbols and nomenclature as Chemists and Electrical Engineers have done. The Committee on Notation of the Chamber of Delegates to the International Electrical Congress, which met at Chicago in 1893, recommended a set of "Symbols for Physical Quantities and Abbreviations for Units," which have ever since been (almost) universally adopted throughout the world by Electricians.* This at once enables the results of certain new or corroborative investigations and formulæ, which may have been made and printed anywhere, to be clearly understood anywhere else, without having to specially interpret the precise meaning of each symbol letter.

In the following list of symbols, abbreviations and index letters, the *first* letter of the chief noun or most important word has been used to indicate the same. Where it appeared necessary, the *first* letter or letters of the adjectival substantive or qualifying words have been added, either as a following or as a subscript or suffix letter or letters. For certain specific quantities, ratios, coefficients and angles, small Greek letters have been used, and I have added to this list the complete Greek alphabet, since it may be refreshing to the memory of some to again see and read the names of these letters, which were no doubt quite familiar to them when at school.

*These "Symbols for Physical Quantities and Abbreviations for Units" will be found printed *in full* in the form of a table at the commencement of Munro and Jamieson's *Pocket-Book of Electrical Rules and Tables*. If a similar recommendation were authorised by a committee composed of delegates from the chief Engineering Institutions, it would be gladly adopted by "The Profession" in the same way that the present work of "The Engineering Standards Committee" is being accepted.

TABLE OF MECHANICAL ENGINEERING QUANTITIES, SYMBOLS, UNITS
AND THEIR ABBREVIATIONS.

(As used in Vols. I. to V. of Prof. Jamieson's "Applied Mechanics.")

Quantities.	Symbols.	Defining Equations.	Practical Units.	Abbreviations of the Practical Units.
FUNDAMENTAL.				
Length, . . .	L, l	...	{ Yard, . . .	yd.
			{ Foot, . . .	ft.
			{ Inch, . . .	in.
Mass, . . .	M, m	...	{ Pound, . . .	lb.
			{ Second, . . .	s.
Time, . . .	T, t	...	{ Minute, . . .	m.
			{ Hour, . . .	h.
GEOMETRIC.				
Surface, . . .	S, s	$S = L^2$	{ Square foot, . . .	sq. ft.
			{ Square inch, . . .	sq. in.
Volume, . . .	V	$V = L^3$	{ Cubic foot, . . .	cb. ft.
			{ Cubic inch, . . .	cb. in.
			{ Degree, . . .	1°
Angle, \angle . . .	$\left\{ \begin{array}{l} \alpha, \beta \\ \theta, \phi \end{array} \right\}$	$\alpha = \frac{\text{arc}}{\text{radius}}$	{ Minute, . . .	1'
			{ Second, . . .	1"
			{ Radian = $\frac{180^\circ}{\pi}$. . .	rn.
MECHANICAL.				
Velocity, . . .	v	$v = \frac{L}{T}$	Foot per second, . . .	$\frac{\text{ft.}}{\text{s.}}$
Angular velocity, . . .	ω	$\omega = \frac{v}{L} = \frac{\theta}{t}$	{ Revs. per second, . . .	r.p.s.
			{ Revs. per minute, . . .	r.p.m.
			{ Radians per second, . . .	$\frac{\text{rad.}}{\text{s.}}$
Acceleration, . . .	a, g	$a = \frac{v}{T}$	Foot per sec. per sec. . .	$\frac{\text{ft.}}{\text{s}^2}$
Force, . . .	F, f	...	{ Pound weight (gravitational unit), . . .	lb. wt. (or lb.)
	W, w	$F = Ma$	{ Poundal (absolute unit), . . .	pdl.
Pressure (per unit area), . . .	p	$p = \frac{F}{s}$	Pound per sq. inch, . . .	lb. \square''
Work, . . .	(Wh)	$Wh = FL$	Foot-pound, . . .	ft.-lb.
Potential energy, . . .	E_p	$E_p = Wh$	Foot-pound, . . .	ft.-lb.
Kinetic energy, . . .	E_k	$E_k = \frac{Wv^2}{2g}$	Foot-pound, . . .	ft.-lb.
Power or activity, . . .	HP	$H.P. = \frac{Wh}{T}$	{ Horse power, . . .	H.P.
			{ Ft.-lb. per min., . . .	ft.-lb./m.
			{ Ft.-lb. per sec., . . .	ft.-lb./s.
Moment of inertia, . . .	I	$I = Mk^2$	lb.-ft. ²
Density, . . .	ρ	$\rho = \frac{M}{V}$	{ Pound per cb. ft., . . .	$\frac{\text{lb.}}{\text{ft.}^3}$
			{ Pound per cb. in., . . .	$\frac{\text{lb.}}{\text{in.}^3}$

OTHER SYMBOLS AND ABBREVIATIONS IN VOLS. I. TO V.

A for Areas.	x, y, z for Unknown quantities.
B, b ,, Breadths.	Z ,, Modulus of section.
C, c, k ,, Constants, ratios.	Z_t ,, ,, tension.
c.g. ,, Centre of gravity.	Z_c ,, ,, compression.
D, d ,, Diameters depths, deflections.	
D_1, D_2, D_3 ,, Drivers in gearing.	Δ, δ, d for Differential signs which are prefixed to another letter; then the two together represent a very small quantity.
E ,, Modulus of elasticity.	ϵ, e ,, Represents base of Napierian Logs = 2.7182; for example, $\log_e 3 = 1.1$.
e ,, Velocity ratio in wheel gearing.	η ,, Efficiency.
F_1, F_2, F_3 ,, Followers in gearing.	λ ,, Length ratio of ship to model.
f, f_t ,, Forces of shear and tension.	μ ,, Coefficient of friction.
H, h ,, Heights, heads.	π ,, Circumference of a circle \div its diameter.
H.P., h.p. ,, Horse-power.	ρ ,, Radius of curvature, radian.
B.H.P. ,, Brake horse-power.	
E.H.P. ,, Effective ,,	Σ for Symbol for sum total of a number of quantities.
I.H.P. ,, Indicated ,,	\int_0^x ,, Sign of integration or summation between limits 0 and x .
k ,, { Radius of gyration, or, Coef. of discharge in hydraulics.	\sim ,, Sign for the difference between two quantities.
N, n ,, Numbers—e.g., number of revs. per min., number of teeth, &c.	\square ,, Sign for square—e.g., $10 \square'' = 10$ square inches.
P, Q ,, Push or pull forces.	— ,, Sign over two letters, \overline{PQ} for a force acting from P to $\rightarrow Q$, means that they represent a vector quantity, which has (1) magnitude, (2) direction, (3) sense.
$R_1 R_2$,, Reactions, resultants, radii, resistances.	\supseteq ,, Sign for equal to or greater than.
s ,, { Seconds, space, surface. Displacement, distance.	\leq ,, Sign for equal to or less than.
SF ,, Shearing force.	
TM ,, Torsional moment.	
TR ,, Torsional resistance.	
BM ,, Bending moment.	
MR ,, Moment of resistance.	
RM ,, Resisting moment.	
T_d, T_s ,, Tensions on driving and slack sides of belts or ropes, &c.	
W_L, W_T, W_U ,, Lost, total, and useful work.	

GREEK ALPHABET.

A	α	Alpha.	I	ι	Iota.	P	ρ	Rho.
B	β	Beta.	K	κ	Kappa.	Σ	σ or ς	Sigma.
Γ	γ	Gamma.	Λ	λ	Lambda.	T	τ	Tau.
Δ	δ	Delta.	M	μ	Mu.	Υ	υ	Upsilon.
E	ϵ	Epsilon.	N	ν	Nu.	Φ	ϕ	Phi.
Z	ζ	Zeta.	Ξ	ξ	Xi.	X	χ	Chi.
H	η	Eta.	O	\omicron	Omicron.	Ψ	ψ	Psi.
Θ	θ	Theta.	Π	π	Pi.	Ω	ω	Oméga.

APPLIED MECHANICS.

THE PRINCIPLE OF WORK AND ITS APPLICATIONS; FRICTION, POWER TESTS, WITH EFFICIENCIES OF MACHINES, MOTION AND ENERGY.

LECTURE I.

CONTENTS.—Definition of Applied Mechanics—Definition of Matter—Definition of Force—Unit of Force—Definition of Absolute Unit of Force—Definition of British Absolute Unit of Force—Definition of Gravitation Unit of Force—Relation between the Gravitation and Absolute Units of Force—Definition of Work—Definition of British Unit of Work—Definition of British Absolute Unit of Work—Example I.—Work done by a Force Acting Obliquely to the Direction of Motion—Example II.—Propositions I. and II.—Examples III. and IV.—Questions.

Applied Mechanics is that branch of applied science which (1) explains the principles upon which machines and structures are made, (2) how they act, and (3) how their strength and efficiency may be tested and calculated.

In this treatise, we shall be chiefly concerned with the application of mechanical laws and principles to the determination of the equilibrium of machines, when acted on by forces; the transmission of power by machines and fluids; the stresses in, and the stability of, structures in general.

Although the student is expected to possess an elementary knowledge of the subject as far as it is treated in the author's *Manual on Applied Mechanics*, yet it is necessary to define, and explain briefly, in their respective places, the more elementary terms which will be used in this book. The student should not content himself with merely learning by rote the definitions herein given, but he should first get a clear understanding of the whole meaning of the things defined, and then endeavour to acquire the facility of defining the terms in his own words.

DEFINITION.—Matter is anything which can be perceived by our senses, or which can exert, or be acted on by, force.

What matter is in itself we know not, we only know it by

its properties, its effects on other pieces of matter, and on our senses.

DEFINITION.—Force is that which produces, or tends to produce, motion or change of motion in the matter upon which it acts.

So far as we are concerned we shall consider that mechanical force acts on matter either by a “push,” “thrust,” or “pressure,” or by a “pull.”

Unit of Force.—Since force is a measurable quantity, we must have a unit of force by which to measure other forces. In this country two units of force are in use, called, respectively, the Gravitation Unit and the Absolute Unit. The gravitation unit of force is adopted by engineers; and is used in the solution of most Statical problems in Theoretical Mechanics. The absolute unit of force is generally adopted in physical investigations; and, also, for convenience in most Kinetic problems in Theoretical Mechanics.

The distinction between these two units of force will be understood from the following definitions:—

DEFINITION.—An Absolute Unit of Force may be defined as that force which, acting for unit time on unit mass, imparts to it unit velocity.

This is the general definition of an absolute unit of force, and by substituting proper units for time, mass, and velocity we get the various absolute units of force for any system in which time, mass, and length are adopted as the fundamental units. An absolute unit of force is, therefore, quite independent of the various values of gravity at different latitudes and of all other variable forces. In other words, it is an independent and invariable unit of force.

If the units of time, mass, and velocity be the second, pound, and foot per second respectively, we then get the following:—

DEFINITION.—The British Absolute Unit of Force, called the Poundal, is that force which, acting for one second on a mass of one pound, imparts to it a velocity of one foot per second.

DEFINITION.—Our Gravitation Unit of Force, called the Pound, is the force required to support a mass of one pound avoirdupois against the attractive force of gravity at Greenwich sea level.

Hence, the magnitude of a force, in gravitation units, is

numerically equal to the mass in pounds which it is just capable of supporting against gravity at Greenwich sea level.

Since all places on the earth's surface (even when at the same level) are not at the same distance from the centre of mass of the earth; since the earth is not of uniform density, and since the effect of centrifugal force due to the earth's rotation varies with the latitude (being greatest at the equator and zero at the poles), it is evident that the weight of a pound of mass will vary with the locality. It is less at places near the equator than at places near the poles. For this reason, then, physicists have adopted the *Absolute* or *Invariable Unit* when dealing with problems in which the results are to be independent of locality and show a high degree of accuracy.

Relation between the Gravitation and Absolute Units of Force.
—The symbol g may be defined as the number of feet per second by which the attractive force of gravity would increase, during every second, the velocity of a body falling freely *in vacuo* near the earth's surface. The value of g is about 32.2 at the latitude of London. Clearly, then, the gravitation unit is g times the absolute unit.

Hence, A force of one pound = g poundals.

Or, A force of one poundal = $\frac{1}{g}$ pound.

DEFINITION.—Work is said to be done by a force when it overcomes a resistance through a distance along the line of action of the resistance.

Hence, if a force act upon matter and causes relative motion of its atoms, or relative change of motion between one body and another, then the force is said to do work.

In the mechanical sense of the term, *work* implies two things—(1) that some *effort* has been exerted or a resistance overcome; (2) That something is moved or a *displacement* takes place. Hence the two elements of work are effort (or resistance) and motion (or displacement).*

* The word "*effort*" is a very expressive term, implying the positive or active aspect of force; whereas the word "*resistance*" naturally conveys to one the negative or opposing aspect of force. By Newton's Third Law action and reaction (or effort and resistance) are equal and opposite, hence the terms "*effort and action*" or "*resistance and re-action*" are variously used in problems to denote one and the same force, according to the way in which the problem is viewed.

The work done by a force is measured by the product of the *numerical value* of the force and the *numerical value* of the displacement along its line of action.

DEFINITION.—The British Unit of Work, called the Foot-pound (ft.-lb.) is the work done when a force of one pound acts through a distance of one foot along its line of action.

DEFINITION.—The British Absolute Unit of Work, called the Foot-poundal (ft.-pdl.), is the work done when a force of one poundal acts through a distance of one foot along its line of action. Hence, the gravitation unit of work is equal to g absolute units. Therefore, to convert x ft.-lbs. into ft.-pdl., multiply the former number by g , since 1 lb. = g poundals, and, if we multiply both sides of this equation by x feet, we get:—

$$(x) \text{ ft.-lbs.} = (x \times g) \text{ ft.-pdl.}$$

Let $x = 7$, then $(7 \times 32) = 224$ ft.-pdl. = number of foot-poundals in 7 ft.-lbs. of work.

Let P = Force in lbs. (supposed to be constant or uniform).

„ L = Displacement of force in ft. (this displacement being along the line of action of the force).

Then, from the above definitions, we get:—

$$\text{Work done} = (P \times L) \text{ ft.-lbs.}$$

The work done by a variable force will be considered in our next Lecture. In any case, if P represents the mean or average force during the displacement L , then $P \times L$ is the work done.

EXAMPLE I.—The bore of a pump is 8 inches, and the vertical lift is 54 yards, find the weight of the column lifted. If the stroke of the pump bucket be 9 feet, and the number of strokes 8 per minute, find the work done in one hour.

ANSWER.—Diameter of bucket = 8 ins. = $\frac{2}{3}$ ft.; vertical lift or head of water = $54 \times 3 = 162$ ft.; stroke of bucket = 9 ft.; number of strokes of bucket = $8 \times 60 = 480$ per hour.

(1) To find the weight of the column lifted.

$$\text{Volume of water lifted} = \text{Volume of column} = \frac{\pi}{4} d^2 l.$$

$$\text{„ „} = .7854 \times \left(\frac{2}{3}\right)^2 \times 162 = 56.55 \text{ cub. ft.}$$

$$\therefore \text{Weight of column lifted} = 56.55 \times 62.5 = 3,535 \text{ lbs.}$$

(2) To find the work done per hour.

Work done in one stroke = Total pressure on bucket \times stroke.

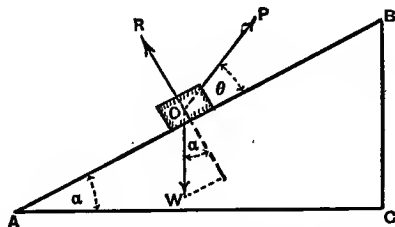
$$= 3,535 \times 9.$$

$$\therefore \text{Work done per hour} = 3,535 \times 9 \times 480.$$

$$= 15,271,200 \text{ ft.-lbs.}$$

Work done by a Force Acting Obliquely to the Direction of Motion.—Referring to the previous definition of work, the student will notice that the factor L , in the product $P \times L$, means the displacement of the point of application of the force, P , *along its line of action*. In many cases the line of action of the force is *oblique* to the line of motion, and we now proceed to show how the work done is measured in such cases.

Consider the case of a body being drawn along a *smooth* inclined plane, AB , by an effort, P , whose line of action is inclined at an angle, θ to AB .



WORK DONE BY A FORCE ACTING OBLIQUELY.

Now, from elementary principles we know that P can be resolved into two components at right angles to each other. One ($P \cos \theta$) in the direction AB , and the other ($P \sin \theta$) at right angles to AB . The point of application, O , of P , moves in a direction parallel to AB , and, hence, by the definition just referred to, the latter component ($P \sin \theta$) *does no work*. The only effect of this perpendicular or *normal* component is to diminish the pressure between the body and the plane AB . Hence, the only part of P which is effective in causing motion is the component ($P \cos \theta$) parallel to AB .

Let the body be displaced from A to B .

$$\text{Then, Work done} = P \cos \theta \times AB = P \times AB \cos \theta.$$

But, $AB \cos \theta$ is the length of the *projection* of the displacement, AB , on the line of action of the effort P , or, what is the

same thing, it is the length of the projection of the displacement on a line *parallel* to the line of action of P.*

If we consider the resistance to motion instead of the effort, we get :—

Work done = component of W parallel to AB \times displacement, AB.

$$,, \quad ,, = W \sin \alpha \times AB.$$

$$,, \quad ,, = W \times AB \sin \alpha.$$

$$,, \quad ,, = W \times BC.$$

Here, again, BC is the length of the projection of AB on the direction or line of action of the resistance, W.

Hence, we have the following statement, which is often useful :—

The work done by a force is equal to the product of the force into the length of the projection of the displacement on the line of action or direction of the force.

EXAMPLE II.—A body is dragged along a floor by means of a cord which makes a constant angle of 30° with the floor. The tension in the cord is 10 lbs., weight of body 30 lbs. Find (1) the work done in drawing the body 10 feet along the floor; and (2) the pressure between the body and the floor.

ANSWER.—Here $P = 10$ lbs.; $W = 30$ lbs.; $\theta = 30^\circ$; $L = 10$ ft.

Resolving P into two components at right angles to each other; one in the direction of motion, and the other perpendicular to it, we get :—

$$\text{Horizontal component} = P \cos \theta.$$

$$\text{Vertical component} = P \sin \theta.$$

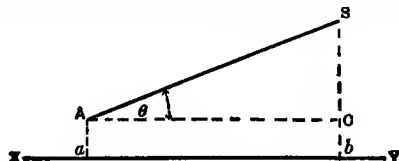
Hence, (1)

$$\text{Work Done} = P \cos \theta \times L.$$

$$,, \quad ,, = 10 \times \cos 30^\circ \times 10.$$

$$,, \quad ,, = 100 \times \frac{\sqrt{3}}{2} = 86.6 \text{ ft.-lbs.}$$

* Let AB and XY, be any two lines inclined to each other at an angle, θ . From A and B draw perpendiculars Aa, Bb to XY. Then ab is called the



ORTHOGONAL PROJECTION.

orthogonal projection of line AB on line XY, and clearly $ab = AB \cos \theta$. In the text the term *projection* is to be understood as *orthogonal projection*.

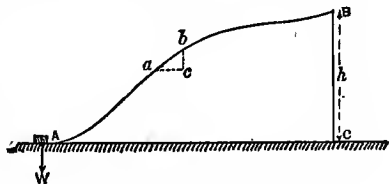
(2) The pressure between the body and the floor is equal to the weight of the body diminished by the vertical or normal component of P .

$$\begin{aligned} \therefore \text{ Pressure between body and floor} &= W - P \sin \theta. \\ &= 30 - 10 \times \sin 30^\circ. \\ &= 30 - 10 \times \frac{1}{2} = 25 \text{ lbs.} \end{aligned}$$

PROPOSITION I.—The work done in lifting a body is independent of the path taken.

When a body of weight, W , is lifted through a vertical height, h , the work done is simply $W h$, and is quite independent of the path described by the body in arriving at its new position.

Suppose the body to be translated from A to B along any route, $A a b B$. Consider the work done in moving the body from a to b , these two points being taken so near to each other that the part of the curve, $a b$, lying between them may be regarded as a straight line. Through a draw $a c$ horizontal and meeting a vertical through b at the point c . Then $a b c$ is a small triangle, and since the resistance overcome is simply that of the weight, W , acting vertically downwards, we get:—



WORK DONE IS INDEPENDENT OF THE PATH TAKEN.

$$\text{Work done from } a \text{ to } b = W \times b c.$$

By dividing the whole path, $A B$, into a great number of parts such as $a b$, we get for total displacement, $A B$:—

$$\text{Work done} = W \times \Sigma b c,$$

where $\Sigma b c$ denotes the sum of all such small vertical distances like $b c$.

$$\text{But,} \quad \Sigma b c = B C = h.$$

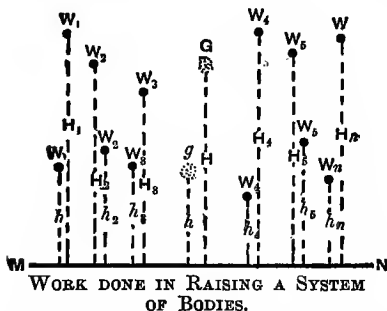
$$\therefore \quad \text{Work done} = W \times h.*$$

PROPOSITION II.—The work done in raising a body or system of bodies is equal to the total weight raised multiplied by the

* This result might have been deduced at once by assuming the results just previously obtained for the case of the inclined plane, by observing that $B C$ is equal in length to the projection of the displacement, $A a b B$, on the direction of the resistance, W . This being a more general case than the one cited, we have thought it better to give an independent proof.

vertical height through which the centre of gravity of the body or system of bodies has been raised.

Suppose we have a number of weights, W_1, W_2, W_3 , &c., at different heights, h_1, h_2, h_3 , &c., respectively, above a given plane, MN . Let the *c.g.* of the system be at g , at a height, h , above MN .



If all the weights be now lifted into different positions, so that the heights above MN are H_1, H_2, H_3 , &c., respectively, and their *c.g.* at a height H . Then,

$$\text{Total work done} = W_1(H_1 - h_1) + W_2(H_2 - h_2) + W_3(H_3 - h_3) + \&c.$$

But, by a property of the *c.g.* we know that

$$W_1 H_1 + W_2 H_2 + W_3 H_3 + \dots = (W_1 + W_2 + W_3 + \dots) H.$$

And,

$$W_1 h_1 + W_2 h_2 + W_3 h_3 + \dots = (W_1 + W_2 + W_3 + \dots) h.*$$

Subtracting the latter equation from the former, we get:—

$$\begin{aligned} W_1(H_1 - h_1) + W_2(H_2 - h_2) + W_3(H_3 - h_3) + \dots \\ = (W_1 + W_2 + W_3 + \dots)(H - h), \end{aligned}$$

$$\therefore \text{Total work done} = (W_1 + W_2 + W_3 + \dots)(H - h) = W(H - h).$$

Where, $W = W_1 + W_2 + W_3 + \&c.$

And, $H - h$ = vertical height through which the *c.g.* of the system has been raised.

Although we have taken a system of disconnected weights in proving the above proposition, the student will clearly perceive that the result arrived at is true generally, whatever form the material may have.

The following simple examples will show the application of the two preceding propositions:—

EXAMPLE III.—A uniform beam, 20 ft. long, and weighing 30 cwts., is lying on the ground. Find the work done in raising it into a vertical position by turning it about one end.

* The student will readily see that these results are arrived at by taking the moments of the weights about the plane, MN , and then applying the "principle of moments."

ANSWER.—The centre of gravity of the beam is 10 ft. from either end, and, during the operation of lifting the beam, this point will describe an arc, which is the quarter of the circumference of a circle whose centre is at the end of the beam in contact with the ground. The vertical height through which the *c.g.* is raised is, therefore, 10 ft.

Hence, by the two preceding propositions,

Work done = whole weight of beam \times height through which its *c.g.* is raised.

$$= (30 \times 112) \times 10.$$

$$= 33,600 \text{ ft.-lbs.}$$

EXAMPLE IV.—A cistern 22 ft. long, 14 ft. broad, and 12 ft. deep, has to be filled with water from a well 7 ft. in diameter. The vertical height of the bottom of the cistern above the free surface of the water in the well is 100 ft. when the operation of filling the cistern is commenced. Water flows into the well at the rate of 462 cubic ft. per hour. Find the work done in filling the cistern, supposing 30 minutes are required for the operation.

ANSWER.—During the operation of filling, the surface of the water in the well will fall, say x ft., from *E F* to *H K*.

The volume of water taken from the well = volume of water *E F H K* + volume of water run in during the operation.

But, Volume of water taken from well

$$= \text{volume of tank } A B C D$$

$$= 22 \times 14 \times 12 \text{ (cub. ft.)}$$

Volume of water represented by *E F H K*

$$= \frac{\pi}{4} d^2 x = \frac{11}{14} \times 7^2 \times x \text{ (cub. ft.)}$$

Volume of water run in in 30 minutes

$$= \frac{462}{2} = 231 \text{ (cub. ft.)}$$

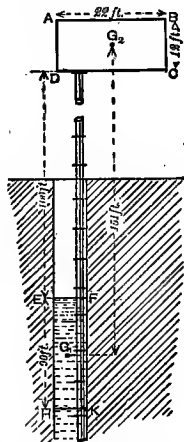
$$\therefore \frac{11}{14} \times 7^2 \times x + 231 = 22 \times 14 \times 12$$

$$\therefore x = 90 \text{ ft.}$$

Clearly, then, the *c.g.* of the water has been raised from G_1 to G_2 , or through a height of $45 + 100 + 6 = 151$ ft.

\therefore Work done in filling cistern

$$= (22 \times 14 \times 12 \times 62\frac{1}{2}) \times 151 = 34,881,000 \text{ ft.-lbs.}$$



WORK DONE IN
FILLING A CISTERN.

LECTURE I.—QUESTIONS.

1. Define the terms *force* and *work done by a force*. What are the units of force and work as adopted by engineers in this country? State the relations between the gravitation and absolute units of force and work, and say why the latter units are so desirable for many scientific purposes.

2. A punching machine is provided with a flywheel and driven by an engine at such a rate that two holes are punched in three minutes. The plate operated on is 1 inch thick, and it is estimated that a mean pressure of 69 tons is exerted through the space of 1 inch. Find the average work done per minute by this machine. *Ans.* 8,586·6 ft.-lbs.

3. A body weighing 100 lbs. is pushed along a horizontal plane by a force of 25 lbs., the direction of which makes an angle of 45° with the plane. Find the work done in moving the body through a distance of 100 feet, and the pressure between the body and the plane. If the direction of the force be reversed, so that it now becomes a pull, find the work done during a displacement of 100 feet, and the pressure between the body and the plane. *Ans.* (1) 1,767 ft.-lbs.; 117·67 lbs. (2) 1,767 ft.-lbs.; 82·33 lbs.

4. Find the work done in turning a cubical block of stone about one of its edges until the diagonals of its end faces are vertical. Length of edge of cube, $4\frac{1}{2}$ ft., *s.g.*, 2·5. *Ans.* 13,242 ft.-lbs.

5. A cistern 22 ft. long, 10 ft. broad, and 8 ft. deep, has to be filled with water from a well 8 ft. in diameter and 40 ft. deep. Supposing no water to flow into the well during the operation of filling the cistern, ascertain how far the surface of the water in the well is depressed, and the work done in filling the cistern when the bottom of the latter is 36 ft. above the free surface of the water in the well at the beginning of the operation. *Ans.* 35 ft.; 6,325,000 ft.-lbs.

LECTURE I.—A.M.INST.C.E. EXAM. QUESTIONS.

1. Taking the diameter of the earth as 8,000 miles, calculate the work in foot-pounds required to remove a stone weighing 1 lb. to an infinite distance from the earth's surface. *Ans.* $21\cdot12 \times 10^6$ ft.-lbs.

2. A pump is delivering water into a boiler in which the pressure is 120 lbs. per square inch above atmospheric pressure. Find the work done in foot-lbs. per lb. of water delivered to the boiler. *Ans.* 276 foot-lbs.

3. Calculate the work required to raise 80 gallons of water through a height of 20 feet. *Ans.* 16,000 foot-lbs.

N.B.—See *Appendices B and C* for other questions and answers. This note refers to all the Lectures.

LECTURE II.

CONTENTS.—Graphical Representation of Work Done—Diagram of Work for any Varying Force—Case I., When the Force varies directly as the displacement—Examples I., II., III., and IV.—Case II., When the Force varies inversely as the displacement—Boyle's Law—Proposition—Work Done by a Gas Expanding according to Boyle's Law—Example V.—Indicator Diagrams—Rate of Doing Work—Definition of Power or Activity—Definition of Horse-Power—Example VI.—Useful and Lost Work—Definition of Efficiency—Table of Efficiencies—Examples VII. and VIII.—Questions.

Graphical Representation of Work Done.—We have already seen that work is the product of two factors—force and displacement. Now a force can be completely represented by a straight line, and so also can a displacement. Since an area is of two dimensions, it follows at once, that work done can be represented by an area. In our elementary manual on Applied Mechanics, we have shown how to represent by diagrams, the work done for several simple cases. For a uniform force the diagram of work is a rectangle; for a uniformly increasing or uniformly decreasing force the diagram will be triangular or trapezoidal in shape. The shape of the diagram will, however, depend on the manner in which the force varies with the displacement.

A correct diagram of work must fulfil the following conditions :—

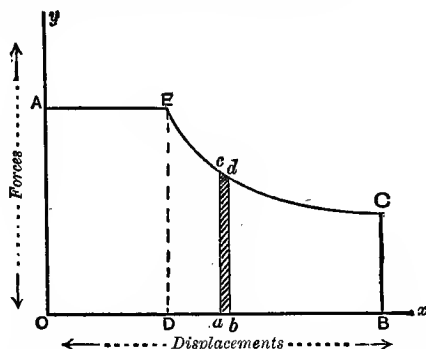
- (1) *Its area must represent the work done.*
- (2) *It must show to the eye the manner in which the force varies in magnitude during the displacement.*

Diagram of Work for any Varying Force.—We shall now show that, if the force during any given displacement be represented in magnitude by the ordinates of the curve, and the displacement by the corresponding abscissæ, the work done will be represented by the area of the figure enclosed between the curve, the initial and final ordinates, and the axis of x .

Let Ox , Oy be rectangular axes; Ox being the axis along which displacements are to be set off, and Oy the axis along which forces are plotted.

Suppose the force at the beginning of the motion to be represented by the ordinate, OA , and at the end of the motion by BC , then AEC is called the *curve of resistance*. At any intermediate point, such as a , the force or resistance will be represented by

the ordinate, $a c$. The work done during the displacement, $O B$, will be represented by the area, $O A E C B$.



GRAPHICAL REPRESENTATION OF WORK DONE BY A VARYING FORCE.

For, suppose the force to be uniform during the displacement, $O D$, then:—

$$\left. \begin{array}{l} \text{Work done during} \\ \text{displacement } O D \end{array} \right\} = \text{Area of rectangle } O A E D.*$$

We have now to show that the work done during the displacement, $D B$, is represented by the area, $D E C B$.

Take any two ordinates, $a c$, $b d$, indefinitely near to each other. The lengths of these ordinates represent the magnitude of the forces at the points a and b respectively. Now, since the ordinates are indefinitely near together, the difference in their lengths will be indefinitely small. In that case $a c d b$ may be considered a rectangle (its breadth being indefinitely small). Hence, the work done during the infinitely small displacement, $a b$, will be represented by the small rectangular strip, $a c d b$. By dividing the displacement, $D B$, into an infinite number of indefinitely small portions, such as $a b$, and drawing the ordinates at these points, an infinite number of narrow rectangles will thereby be obtained. Hence, it is clear that the work done during the displacement, $B D$, is represented by the sum of these elementary areas,

i.e., The Work done during displacement $D B = \text{Area } D E C B$;

\therefore Total work done during displacement $O B = \text{Area } O A E C B$.

* The sign (=) is here used as an abbreviation of the words "is represented by," and must not be employed in its usual sense as meaning "is equal to." The text will enable the student to attach the proper meaning to the sign used.

Two particular cases of work done by varying forces will now be considered.

CASE I.—When the force varies directly as the displacement.

When we stretch or compress a piece of any *solid elastic material*—e.g., a helical or spiral spring, or a bar of iron or steel—the resistance offered is directly proportional to the extension or compression produced, when these are small compared with the length of the body. Thus, if a force of 10 lbs. be required to stretch a spiral spring 1 inch, then a force of 30 lbs. will be required to stretch the same spring 3 inches, and so on.

We may state this law thus:—

Force \propto Displacement.

Or,

$$P \propto L$$

$$\therefore P = cL; \text{ or } \frac{P}{L} = c,$$

where c is some constant quantity depending on the nature of the material.

Hence, if P is the force required to stretch or compress the material by an amount L , and p the force required to stretch or compress the material by an amount l , then

$$P : p = L : l.$$

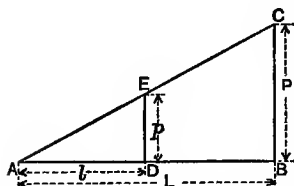


DIAGRAM OF WORK WHEN
THE FORCE VARIES AS
THE DISPLACEMENT.

We shall now show that in the diagram of work for this case, the line of resistance is a straight line.

Set out AB to represent the displacement L , and AD to represent l .

Let the ordinate BC represent P . Join AC , and through D draw the ordinate DE . Then DE will represent p .

By similar triangles,

$$DE : BC = AD : AB$$

$$\text{i.e., } \left. \begin{array}{l} DE : P = l : L \\ p : P = l : L \end{array} \right\} \therefore DE = p.$$

Again, the areas of the triangles ADE , ABC represent the work done during the displacements AD and AB respectively.

For,

$$\left. \begin{array}{l} \text{Work done during} \\ \text{displacement AD} \end{array} \right\} = \text{area of } \triangle ADE = \frac{1}{2} DE \times AD$$

\therefore

$$\left. \begin{array}{l} \text{Work done during} \\ \text{displacement } l \end{array} \right\} = \frac{1}{2} pl.$$

Similarly,

$$\left. \begin{array}{l} \text{Work done during} \\ \text{displacement L} \end{array} \right\} = \frac{1}{2} PL.$$

Also,

$$\left. \begin{array}{l} \text{Work done during} \\ \text{displacement DB} \end{array} \right\} = \text{area of trapezoid DBCE.}$$

$$= \frac{1}{2} (BC + DE) \times DB.$$

$$= \frac{1}{2} (P + p) (L - l).$$

The above results are true whether the force uniformly increases or uniformly decreases. The following examples will render the above principles clear:—

EXAMPLE I.—Show, by a diagram or otherwise, how the work done in stretching an elastic spring is obtained; 5 ft.-lbs. of work are required to stretch a spiral spring 3 inches; what force in lbs. will be required to stretch the same spring $6\frac{3}{4}$ inches?

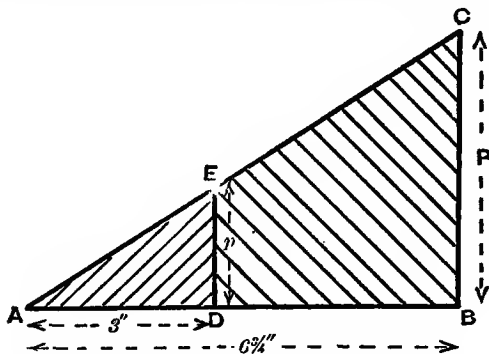


DIAGRAM OF WORK IN STRETCHING A SPRING.

ANSWER.—Let p , P denote the forces required to stretch the spring 3 inches and $6\frac{3}{4}$ inches respectively.

Then, area of $\triangle ADE$ represents the work done in stretching the spring from A to D.

EXAMPLE III.—A chain weighing 2 lbs. per foot passes over a fixed smooth pulley, so that 14 feet hangs over on one side and 6 feet on the other. Show by a diagram the work which will be done in pulling round the wheel until the upper end of the chain is 1 foot above the lower end.

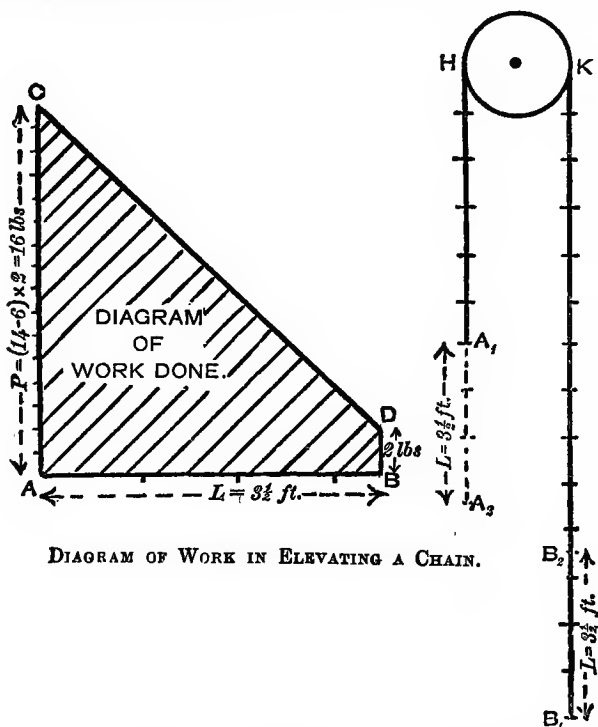


DIAGRAM OF WORK IN ELEVATING A CHAIN.

ANSWER.—Clearly the resistance to be overcome at the beginning of the motion is the weight of the *difference* of the two parts of the chain hanging from the pulley. That is, the initial resistance = weight of a length of $(14 - 6) = 8$ ft. of chain.

$$\therefore \text{Initial resistance} = P = 8 \times 2 = 16 \text{ lbs.}$$

When the upper end of the chain is pulled down so as to be 1 ft. above the lower end, the displacement will be $3\frac{1}{2}$ ft., and the

$$\text{Final resistance} = p = 1 \times 2 = 2 \text{ lbs.}$$

We can now construct the diagram of work. Set off AB to represent the displacement, $= 3\frac{1}{2}$ ft. Make AC represent the initial resistance, $= 16$ lbs., and BD represent the final resistance, $= 2$ lbs. Join DC , then the trapezoid $ABDC$ is the diagram of work, and its area represents the work done.

$$\begin{aligned}\therefore \quad \text{Work done} &= \text{area } ABDC \\ \text{,,} \quad \text{,,} &= \frac{1}{2} (AC + BD) \times AB \\ \text{,,} \quad \text{,,} &= \frac{1}{2} (16 + 2) \times 3\frac{1}{2} \\ \text{,,} \quad \text{,,} &= 31.5 \text{ ft.-lbs.}\end{aligned}$$

Of course, the student will readily see that it is not always necessary to construct a diagram of work before arriving at the answer. All that is necessary to know, is the mean resistance during the displacement. Thus, in the above example, the mean resistance is the arithmetical mean between the initial and final resistances. This, multiplied by the displacement, gives the answer.

EXAMPLE IV.—Four cwts. of material are drawn from a depth of 80 fathoms by a rope weighing 1.15 lbs. per linear foot: how many units of work are expended?

ANSWER.—Here the resistance to be overcome at the beginning of the lift = *whole weight of rope + weight of material*.

$$\text{Whole weight of rope} = (80 \times 6) \times 1.15 = 552 \text{ lbs.}$$

$$\text{Weight of material raised} = 4 \times 112 = 448 \text{ lbs.}$$

$$\therefore \left. \begin{array}{l} \text{Resistance at begin-} \\ \text{ning of lift} \end{array} \right\} = 552 + 448 = 1,000 \text{ lbs.}$$

If we suppose the whole length of rope to be hauled in when the material is brought to the surface, then the resistance to be overcome at end of lift is simply that of the weight of the material to be raised.

$$\therefore \quad \text{Resistance at end of lift} = 448 \text{ lbs.}$$

We can now construct the diagram of work. Set off AB to represent the displacement, $= 80 \times 6 = 480$ ft. Set off AC to represent the initial resistance due to weight of rope, $= 552$ lbs. Make AD represent to the same scale as AC , the resistance due to the weight of the material, $= 448$ lbs. Join CB , then triangle ABC is the diagram of work for the rope or variable part of the load. Complete the rectangle, $ABED$; then $ABED$ is the diagram of work for the material, or constant part of the load. $DOBE$ is the diagram of work for the whole load.

Hence,

$$\begin{aligned}
 \text{Work expended during lift} &= \text{Area } D C B E, \\
 " &= \frac{1}{2} (D C + E B) \times A B, \\
 " &= \frac{1}{2} (1,000 + 448) \times 480, \\
 " &= 347,520 \text{ ft.-lbs.}
 \end{aligned}$$

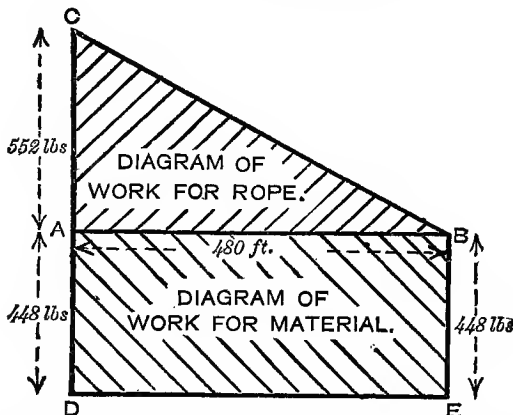


DIAGRAM OF WORK IN ELEVATING A LOAD BY A ROPE OR CHAIN

We could have arrived at the answer quite simply by finding the work done in lifting the rope and material separately, and then adding together the results. Thus :—

$$\begin{aligned}
 \text{Work done in lifting rope} &= \text{weight of rope} \times \text{height through which its c.g. is raised} \\
 " &= 552 \times \frac{1}{2} \times 480 = 132,480 \text{ ft.-lbs.} \\
 \text{Work done in lifting material} &= 448 \times 480 = 215,040 \text{ ft.-lbs.} \\
 \therefore \text{Total work expended} &= 132,480 + 215,040 \\
 " &= 347,520 \text{ ft.-lbs.}
 \end{aligned}$$

CASE II.—When the force varies inversely as the displacement.

We have seen that, when a *solid* elastic material is stretched or compressed within certain limits, the resistance is proportional to the extension or compression. When, however, we compress a gas or allow it to expand, the law expressing the relation between the pressure applied and the expansion or compression produced, is different from that in the case of a solid. By expansion or compression of a gas we mean the increase or decrease produced in its *volume*.

BOYLE'S LAW.—The pressure of a fixed mass of a perfect gas, at a constant temperature, varies inversely as the volume it occupies.*

Let P = absolute pressure, or elastic force, of gas *per square foot*.

„ V = volume of gas *in cubic feet*;

Then, $P \propto \frac{1}{V}$.

Or, $P V = c$, a constant.

The student should carefully note that Boyle's Law is true only for perfect gases, and, also, that the temperature must remain constant throughout the changes of volume. Boyle's Law is very nearly true for dry atmospheric air, and may be applied to most other gases when these are not near their points of liquefaction.

The value of the constant, c , for a given mass, depends on the nature of the gas under consideration; and also, on the constant temperature maintained. Thus, the constant for air at a temperature 32° F. and a mass of one pound is $(14.7 \times 144 \times 12.34) = 26,214$ ft.-lbs., at atmospheric pressure. Where, 12.34 is the volume in cubic feet of 1 lb. of air at 32° F. and 14.7 lbs. the pressure per square inch.

PROPOSITION.—The work done per unit area on or by a gas during a change of volume is equal to the product of the average pressure per unit area into the change of volume.

Let P = average pressure of gas in lbs. per sq. ft.,

„ V_1 = initial volume of gas in cub. ft.,

„ V_2 = final „ „ „

Then, Work done = $P (V_2 \sim V_1)$ ft.-lbs.

For, suppose we have a cylinder fitted with an air-tight frictionless piston, the area of the latter being A square feet. Let this piston enclose a volume of gas in the cylinder equal to V_1 cubic feet. Now, let the piston move through a distance, L , feet in the cylinder, either by doing work in compressing the gas, or by allowing the gas to do work during its expansion. If the gas

* For an experimental demonstration of this law, and its applications to the steam engine, see the author's works on the "Steam Engine." In all applications of Boyle's Law, *absolute* pressures must be taken. The pressure of the atmosphere may be taken at 14.7, or, roughly, 15 lbs. per square inch absolute.

now occupy a volume, V_2 cubic feet, and the average pressure on the piston during its displacement be P lbs. per square foot, then :—

$$\text{Work done} = P A \times L \text{ (ft.-lbs.)}$$

But,

$$A L = \text{Change of volume of gas.}$$

i.e.,

$$= V_2 \sim V_1 \text{ (cub. ft.)}$$

∴

$$\text{Work done} = P (V_2 \sim V_1) \text{ ft.-lbs.}$$

This result is true whatever be the size and shape of the vessel containing the gas. When the vessel is of uniform cross sectional area, it may be convenient to consider only the displacement of the piston, the total pressure on the piston being taken as the effort or resistance. Examples of this will be given immediately.

Work done by a Gas Expanding according to Boyle's Law.—We are now in a position to be able to find the work done by or on a gas during a change of volume when the change takes place at constant temperature on a constant mass of gas.

Let p_1 = initial absolute pressure,

„ v_1 = „ volume,

„ p_2 = final absolute pressure,

„ v_2 = „ volume.

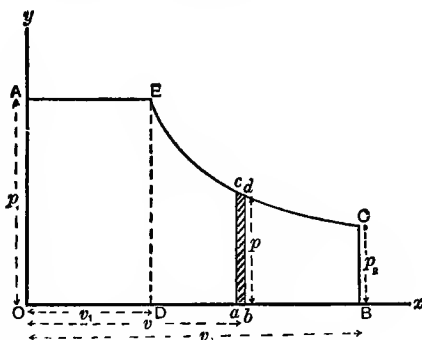


DIAGRAM OF WORK ILLUSTRATING BOYLE'S LAW.

Let OA and OD represent the initial pressure and volume respectively ; BC and OB the final pressure and volume.

Consider the work done during the small increase of volume ab .

$$\text{Let } Oa = v \text{ and } Ob = v + \Delta v$$

$$\text{Then } ab = \Delta v.$$

Let p = mean pressure during increase of volume Δv .

Then, $\text{Work done from } a \text{ to } b = p \Delta v$.

$$\therefore \left. \begin{array}{l} \text{Total work done during} \\ \text{expansion from D to B} \end{array} \right\} = \Sigma p \Delta v.$$

Now, in the limit, when $a b$ is taken infinitely small, p will denote the pressure corresponding to the volume v , and Δv , will, according to the notation of the Calculus be denoted by dv .

$$\therefore \left. \begin{array}{l} \text{Total work done during} \\ \text{expansion from D to B} \end{array} \right\} = \int_{v_1}^{v_2} p dv.$$

But, by Boyle's law,

$$p v = p_1 v_1 = p_2 v_2 = c;$$

$$\therefore p = p_1 v_1 \times \frac{1}{v}.$$

$$\therefore \left. \begin{array}{l} \text{Total work done} \\ \text{during expansion} \\ \text{from D to B} \end{array} \right\} = \int_{v_1}^{v_2} p_1 v_1 \times \frac{dv}{v}$$

$$\begin{array}{ll} \text{"} & \text{"} \\ & = p_1 v_1 \int_{v_1}^{v_2} \frac{dv}{v} \text{ (since } p_1 v_1 = \text{a constant)} \end{array}$$

$$\begin{array}{ll} \text{"} & \text{"} \\ & = \left[p_1 v_1 \log_e v \right]_{v_1}^{v_2} \end{array}$$

$$\begin{array}{ll} \text{"} & \text{"} \\ & = p_1 v_1 (\log_e v_2 - \log_e v_1) \end{array}$$

$$\text{Similarly, " " } = p_1 v_1 \log_e \frac{v_2}{v_1}.$$

$$\left. \begin{array}{l} \text{Total work done} \\ \text{during expansion} \\ \text{from D to B} \end{array} \right\} = p_2 v_2 \log_e \frac{v_2}{v_1}.$$

Where $\log_e \frac{v_2}{v_1}$ is the *Napierian* or *hyperbolic logarithm* of the ratio of the final to the initial volume.* This ratio $\frac{v_2}{v_1}$, is often called the *ratio of expansion*, and is denoted by the letter r . Since the above is true whether the gas be expanded or compressed, we get:—

$$\left. \begin{array}{l} \text{Work done during expansion or} \\ \text{compression between } v_2 \text{ and } v_1 \end{array} \right\} = p_1 v_1 \log_e r.$$

$$\text{Or, " " " } = p_2 v_2 \log_e r.$$

$$\therefore \text{" " " } = c \log_e r.$$

Where c is the constant in the equation $p v = c$.

* The curve $EcdC$ is a *rectangular hyperbola*, the axes Ox , Oy being asymptotes.

The following example will impress the above results more firmly on the mind :—

EXAMPLE V.—Calculate the work done when 10 cub. ft. of air at an initial absolute pressure of 45 lbs. per square inch, is expanded at constant temperature to a volume of 50 cub. ft. Find, also, the final pressure.*

ANSWER.—Here, $P_1 = 45 \times 144 = 6,480$ lbs. per sq. ft.

$$V_1 = 10 \text{ cub. ft.}$$

$$V_2 = 50 \quad ,,$$

$$\therefore \quad r = \frac{V_2}{V_1} = \frac{50}{10} = 5.$$

Then,

$$\text{Work done during expansion} = P_1 V_1 \log_e r$$

$$,, \quad ,, \quad = 6,480 \times 10 \times \log_e 5 \text{ (ft.-lbs.)}$$

Referring to a table of hyperbolic logarithms, we find :—

$$\log_e 5 = 1.6094.$$

$$\therefore \text{Work done during expansion} = 6,480 \times 10 \times 1.6094,$$

$$,, \quad ,, \quad = 104,289.92 \text{ ft.-lbs.}$$

Next, to find the final pressure.

$$\text{Here,} \quad p_2 V_2 = p_1 V_1;$$

$$\therefore \quad p_2 \times 50 = 45 \times 10.$$

$$\text{i.e.,} \quad p_2 = \frac{45 \times 10}{50} = 9 \text{ lbs. absolute per sq. in.}$$

Indicator Diagrams.—As an important application of the diagram of work we may here briefly refer to indicator diagrams obtained from a steam or gas engine. Every engineer knows the importance of obtaining correct diagrams of work done by the steam or gas in the working cylinder of his engine. By an inspection of the cards thus obtained, he is able to detect faults in the working of the engine, which could not be revealed by any other method. For example, from such diagrams he can at once tell whether the valves are properly set; the manner in which the pressure on the piston varies throughout the stroke; the state of the vacuum in the condenser, if it be a steam engine, and a multitude of other facts. He can also calculate the area of the diagram, and thereby deduce the horse-power developed in the cylinder. Lastly, he can

* Expansion at constant temperature is called “*Isothermal*” expansion.

compare this *indicated horse-power* with the *brake horse-power* given out at any point of the machinery during its transmission, and so find the power spent on friction, &c.

Rate of Doing Work.—In the definition and examples of work given in this and the last lecture it will be noticed that no reference was made to the time taken to perform the work. Thus, in Example III., we saw that the work done in raising the 4 cwt. of material and the rope was 347,520 ft.-lbs., and this result is true no matter what time was taken to accomplish it. It did not affect the question of work done whether the material was raised in twenty minutes by the action of men on a windlass, or in one minute by the action of a steam engine. But, if we wish to compare those two agents in respect to the *rate* at which they perform the work, it is clear that this will be in the proportion of 1 : 20. Thus :—

$$\text{Rate of doing work} \dots = \frac{\text{Work done}}{\text{Time taken}};$$

$$\therefore \frac{\text{Rate at which the men work}}{\text{Rate at which the engine works}} = \frac{1}{20} \bigg/ \frac{1}{1} = \frac{1}{20}.$$

Hence, although the *amount* of work done is the same in both cases, yet the *rate* of doing the work is *inversely* as the time taken to do it.

DEFINITION.—Power and Activity are the terms used to denote the rate of doing work.

It is evident, that in order to compare the respective powers of two agents, we must have a standard or *unit of power*. The unit of power adopted in this country is the **Horse-Power**. This unit was first introduced by Watt in estimating the power of his engines, and is still the unit adopted by British engineers.

DEFINITION.—The Unit of Power, called the Horse-power, is the rate of doing work corresponding to 550 ft.-lbs. per second, or 33,000 ft.-lbs. per minute, or 1,980,000 ft.-lbs. per hour.

Although a horse's power was thus defined by Watt, yet no horse is capable of working at the above rate for any length of time. The actual power of a good horse, working for 10 hours a day, is found to be about 22,000 ft.-lbs. per minute instead of 33,000 ft.-lbs. per minute. The term, however, is still retained by engineers, although it is not now used in its original sense.*

* For historical account of this term, see the author's *Elementary Manual on Steam and Steam Engines*, p. 154.

- Let P = Average pressure or effort exerted, in lbs.
 „ L = Displacement of P , in feet.
 „ T = Time taken to perform a given amount of work.
 „ H.P. = Horse-power.

Then, $H.P. = \frac{P \times L}{550 \times T}$, when T is expressed in seconds.

$$" = \frac{P \times L}{33,000 \times T}, \quad " \quad " \quad \text{minutes.}$$

$$" = \frac{P \times L}{1,980,000 \times T}, \quad " \quad " \quad \text{hours.}$$

EXAMPLE VI.—The two cylinders of a locomotive engine are each 17 inches in diameter. Length of stroke, 24 inches. Mean effective pressure of steam on pistons, 80 lbs. per square inch. Diameter of driving wheels, 6 feet. Speed of engine and train, 30 miles per hour. Find the horse-power exerted by engine.

ANSWER.—(1) Find the work done per revolution of driving wheels :—

$$\left. \begin{array}{l} \text{Total effective pressure} \\ \text{on each piston} \end{array} \right\} = P = \frac{\pi}{4} d^2 p,$$

$$" \quad " \quad " = \frac{11}{14} \times 17^2 \times 80 = 18,165.7 \text{ lbs.}$$

Since there are two equal cylinders, and each piston makes two strokes per revolution of the driving wheels, we get :—

$$\left. \begin{array}{l} \text{Total work done per revolu-} \\ \text{tion of driving wheels} \end{array} \right\} = 2 P \times 2 L,$$

$$" \quad " \quad = 2 \times 18,165.7 \times 2 \times 2;$$

$$" \quad " \quad = 145,325.6 \text{ ft.-lbs.}$$

(2) Find the number of revolutions of driving wheel per hour.

$$30 \text{ miles per hour} = 30 \times 5,280 \text{ (ft. per hour).}$$

$$\left. \begin{array}{l} \text{Circumference of} \\ \text{driving wheels} \end{array} \right\} = \pi D = \frac{22}{7} \times 6 \text{ (feet).}$$

$$\therefore \left. \begin{array}{l} \text{Number of revolutions} \\ \text{of driving wheels} \end{array} \right\} = \frac{30 \times 5,280}{\frac{22}{7} \times 6} = 8,400 \text{ (per hour).}$$

$$\therefore \left. \begin{array}{l} \text{Total work done per} \\ \text{hour} \end{array} \right\} = 145,325.6 \times 8,400 \text{ (ft.-lbs.)}$$

$$\therefore \text{H.P. Exerted} = \frac{145,325.6 \times 8,400}{1,980,000} = 616.5.$$

Useful and Lost Work.—Up to this point, we have had no occasion to refer to the relation between the *Useful Work* given out by a working agent, and the *Whole Work Expended*. A machine is erected to perform a given amount of work, which is called the *Useful Work*, but during the working of the machine a considerable part of the whole work expended is absorbed in overcoming frictional resistances, &c., and this work is usually spoken of as the *Lost Work*. The sum of the *Useful Work* and the *Lost Work* is equal to the *Total Work Expended*; or,

$$\text{Total Work Expended} = \text{Useful Work} + \text{Lost Work}.$$

DEFINITION.—The *Efficiency* of a Machine is the ratio of the *Useful Work Done* to the *Total Work expended*.

$$\text{Or,} \quad \text{Efficiency} = \frac{\text{Useful Work Done}}{\text{Total Work Expended}}.$$

Now, the useful work done is always less than the total work expended, hence the efficiency will always be a number less than unity. What is known as the *Percentage Efficiency* is the efficiency, as found above, multiplied by 100. We shall have examples of the efficiencies of several machines later on; but in the meantime it may be instructive to note the efficiencies of a few of the more common machines.

TABLE OF EFFICIENCIES.

NAMES OF MACHINES.	EFFICIENCY.	PERCENTAGE EFFICIENCY.
Wheel and Compound Axle,	·58	58
Simple Screw Jack,	·25	25
Worm and Worm Wheel,	·3 to ·6	30 to 60
Block and Tackle,	·75	75
Weston's Differential Blocks,	·4	40
Hydraulic Ram,	·6	60
Pumps for Draining Mines,	·66	66
Turbine,	·7 to ·8	70 to 80
Overshot Water-Wheel,	·6 „ ·8	60 „ 80
Undershot „ (Common),	·3 „ ·4	30 „ 40
„ „ (Poncelet's),	·6	60
Breast Wheel,	·5 to ·7	50 to 70
Best Compound Steam Engine,	·8 „ ·9	80 „ 90
Gas Engine,	·75 „ ·8	75 „ 80

EXAMPLE VII.—What horse-power is required to lift 3,000 cubic feet of water per hour to a height of 80 feet, supposing $\frac{1}{4}$ of the power to be lost by friction, &c.?

$$\text{ANSWER.—Weight of water raised } \left. \begin{array}{l} \text{every minute} \end{array} \right\} = \frac{3,000 \times 62.5}{60} = 3,125 \text{ lbs.}$$

$$\therefore \text{ Useful work done per minute} = 3,125 \times 80 = 250,000 \text{ ft.-lbs.}$$

Let T.H.P. denote the *theoretical* horse-power required, *i.e.*, the power required when all frictional losses are neglected, and A.H.P. the *actual* horse-power required.

$$\text{Then,} \quad \text{T.H.P.} = \frac{250,000}{33,000} = 7.57.$$

But, according to the question, $\frac{1}{3}$ of the *actual* power is lost in friction, &c.

$$\therefore \quad \text{A.H.P.} - \frac{1}{3} \text{ A.H.P.} = \text{T.H.P.}$$

$$\text{i.e.,} \quad \frac{2}{3} \text{ A.H.P.} = \text{T.H.P.}$$

$$\therefore \quad \text{A.H.P.} = \frac{3}{2} \times 7.57 = 11.36.$$

EXAMPLE VIII.—If there were 4,000 cubic feet of water in a mine, whose depth is 60 fathoms, when an engine of 70 horse-power began to work the pumps, and the engine continued to work for 5 hours before the mine was cleared of the water, find the number of cubic feet of water which had run into the mine per hour, supposing $\frac{1}{3}$ of the power of the engine to be lost in the transmission.

ANSWER.—Let x = number of cub. ft. of water run into the mine in one hour.

Then,

$$\left. \begin{array}{l} \text{Volume of water} \\ \text{pumped per hour} \end{array} \right\} = x + \frac{4,000}{5} = (x + 800) \text{ cub. ft.}$$

$$\therefore \text{ Useful work done } \left. \begin{array}{l} \text{per hour} \end{array} \right\} = (x + 800) \times 62.5 \times (60 \times 6) \text{ ft.-lbs.}$$

Now, since $\frac{1}{3}$ of the power of the engine is lost in the transmission, the remaining $\frac{2}{3}$ will be employed in doing the above work.

$$\therefore \text{ Useful work done by } \left. \begin{array}{l} \text{engine per hour} \end{array} \right\} = \frac{2}{3} \times 70 \times 33,000 \times 60 \text{ (ft.-lbs.)}$$

$$\text{i.e.,} \quad (x + 800) \times 62.5 \times (60 \times 6) = \frac{2}{3} \times 70 \times 33,000 \times 60.$$

$$\text{Or,} \quad x + 800 = \frac{70 \times 88 \times 2}{3} \text{ (cub. ft.)}$$

$$\therefore \quad x = 3,306.6 \text{ cub. ft.}$$

* The method of answering this class of questions is very frequently misunderstood by students. In an example like the above the student is very liable to *increase* the *useful work* by $\frac{1}{3}$ of its amount and then find the H.P. required. The author finds the above method of answering the question appeals more directly to students.

LECTURE II.—QUESTIONS.

1. A spiral spring is stretched through $\frac{1}{2}$ inch by a force of 10 lbs. Find the work done in stretching it through an additional length of 2 inches. Draw the diagram of work done, giving dimensions. *Ans.* 60 inch-lbs.

2. A chain weighing 3 lbs. per foot passes over a fixed smooth pulley, so that 20 feet hangs on one side, and 10 feet on the other. Find the work done in pulling round the wheel until the upper end of the chain is 6 inches above the lower end. Explain clearly the method of setting out the diagram of work in this case and construct it. *Ans.* 74·8 ft.-lbs.

3. A steel wire rope weighing 9 lbs. per fathom is employed to raise 2 tons of material from a depth of 100 fathoms. Find, by calculation, and by a scale diagram of the work, the work done during the lift, supposing the whole length of rope to be wound on the drum at end of lift. Also find the resistance offered at $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ lift respectively. *Ans.* (1) 2,958,000 ft.-lbs., or 1,320·5 ft.-tons; (2) 5,155 lbs.; (3) 4,930 lbs.; (4) 4,705 lbs.

4. Investigate an expression for the work done when a gas is compressed from a volume v_1 , to a volume v_2 , the compression being isothermal—i.e., at a constant temperature. Find the work done in compressing 10 cubic feet of air at a pressure of 15 lbs. per square inch absolute till its pressure is 75 lbs. per sq. in. absolute; given $\log_e 5 = 1·6094$. *Ans.* 34,763 ft.-lbs.

5. Find the work done in exhausting a chamber of 100 cubic feet capacity to $\frac{1}{10}$ of an atmosphere, atmospheric pressure being taken at 14·7 lbs. per square inch absolute. Hyperbolic expansion being assumed. *Ans.* 487,393 ft.-lbs.

6. How is the working power of an agent measured? When is an agent said to work with 1 horse-power? One agent (A) lifts 50 lbs. through 100 feet in 4 minutes; a second agent (B) lifts 2 lbs. through 150 feet in a quarter of a minute; what ratio does A's working power bear to B's? *Ans.* A : B = 25 : 24.

7. The travel of the table of a planing machine cutting both ways is 9 feet, and the resistance to be overcome while cutting is taken at 400 lbs. If the number of double strokes made in one hour be 40, find the horse-power absorbed by the machine. *Ans.* 0·145 H.P.

8. In employing furnace ventilation in a coal mine, there is a furnace at the bottom of a shaft which is estimated to raise 100,000 cubic feet of air at 50° F. through 170 feet in 1 minute. What is the rate at which the furnace does work as estimated in horse-power? *Note.*—A cubic foot of air at 50° F. weighs ·078 lb. *Ans.* 40·18.

9. A pump is worked directly from the ram of a water-pressure engine, the cylinder of which is 6 inches in diameter, that of the pump being 8½ inches. The head of water in the supply-pipe which gives the pressure is 450 feet, and that in the delivery pipe is 150 feet: find the ratio of work done to total work expended. *Ans.* 708 : 1.

10. One thousand cubic feet of water has to be raised to a height of 200 feet per minute: the question is, how many horse-power will it be necessary to employ, supposing that one quarter of the power is lost through friction and other causes? *Ans.* 505 H.P.

11. A builder finds that water accumulates in the space for a foundation at the rate of 1,500 cubic feet per hour. This water has to be pumped to a height of 20 feet. The question is, what amount of power will be required

to keep the said foundation dry, supposing that only 0·6 of the power applied is available for useful effect? *Ans.* 1·58 H.P. nearly.

12. Steam enters a cylinder at 80 lbs. per square inch absolute, and is cut off at $\frac{1}{4}$ of the stroke. Diameter of piston, 40 inches, length of stroke, 5 feet. No of revolutions, 50 per minute. Back pressure, 3 lbs. per square inch absolute. Find the horse-power of the engine, assuming the steam to expand hyperbolically, $\log_e 3 = 1·0985$. *Ans.* 1,009 H.P.

13. Sketch and describe the construction and working of the mechanism, including the driving pulleys, by which the table of a machine for planing metals is moved backwards and forwards. If the travel of the table is 9 feet, and it makes 80 double strokes per hour, cutting both ways, find the resistance to motion if there is 5 horse-power actually expended at the tool. (S. and A. Adv., 1899.) (See my "*Elementary Applied Mechanics*" for complete answer to first part.) *Ans.* 6,875 lbs.

N.B.—See Appendices B and C for other questions and answers.

HYPERBOLIC OR NAPIERIAN LOGARITHMS OF RATIOS OF EXPANSION.

No.	Logarithm.	No.	Logarithm.	No.	Logarithm.	No.	Logarithm.
1	0	3·5	1·2527629	6	1·7917595	8·5	2·1400661
1·25	·2231435	3·75	1·3217559	6·25	1·8325814	8·75	2·1690536
1·5	·4054652	4	1·3862943	6·5	1·8718021	9	2·1972245
1·75	·5596157	4·25	1·4469189	6·75	1·9095425	9·25	2·2246236
2	·6931472	4·5	1·5040773	7	1·9459100	9·5	2·2512918
2·25	·8109303	4·75	1·5581446	7·25	1·9810014	9·75	2·2772673
2·5	·9162907	5	1·6094379	7·5	2·0149030	10	2·3025851
2·75	1·0116009	5·25	1·6582280	7·75	2·0476928	12	2·4849065
3	1·0986124	5·5	1·7047481	8	2·0794414	15	2·7080502
3·25	1·1786549	5·75	1·7491998	8·25	2·1102128	18	2·8903847

LECTURE II.—A.M.INST.C.E. EXAM. QUESTIONS.

1. A steam pump is to deliver 1,000 gallons of water per minute against a pressure of 100 lbs. per square inch. Taking the efficiency of the pump to be 0.70, what indicated horse-power must be provided?

(I.C.E., Feb., 1898.)

2. A chain hanging vertically 820 feet long, weighing 14 lbs. per foot at the bottom, 16 lbs. at the middle, and 18 lbs. per foot at the top, but altering gradually, is wound up. What work is done? (Use squared paper.) (I.C.E., Oct., 1898.)

3. Suppose that town refuse is about $\frac{1}{2}$ ton per unit of population per year. If by burning this refuse steam is supplied to engines which give out 1 actual H.P. hour for every 40 lbs. of refuse burnt, and these engines drive pumps of 85 per cent. efficiency, pumping water to a reservoir; and if, further, this water is then used to drive motors in the city, with a total efficiency of 30 per cent., working on an average 2 hours per working day (of which, say, there are 300 to the year), what would be the average horse-power supplied to each of 500,000 houses in a city of 5,000,000 inhabitants? (I.C.E., Feb., 1899.)

4. A steam engine, regulated by changing boiler-pressure, whose cut-off does not alter, drives a dynamo, and at two steady loads the following results are obtained:—

Indicated Power.	Actual Power Transmitted.	Electrical Power in Kilowatts.	Feed-Water per Hour.
95	82	54	Lbs. 2,400
54	40	25	1,540

What are the probable values when the electrical power is 40 kilowatts? Use squared paper. (I.C.E., Feb., 1899.) (See my "*Elementary Applied Mechanics*" for how to plot values on squared paper. Also, Munro and Jamieson's "*Pocket-Book*" for all *Practical Electrical Units*.)

5. A pump has to force 2,000,000 gallons of water per day through a pipe 12 inches in diameter and 5 miles long to an elevation of 100 feet; find the H.P. required. (I.C.E., Oct., 1901.)

6. Show how to measure work done when the resistance overcome is variable. Find the work done in lifting a spring-loaded safety-valve through 1 inch. Load on valve 135 lbs. per square inch, diameter $3\frac{1}{2}$ inches, initial length of spring 20 inches, length when free 24 inches.

(I.C.E., Oct., 1902.)

7. An elastic string is used to lift a weight of 20 lbs. How much energy must be exerted in raising it 3 feet, supposing the string to stretch 1 inch under a tension of 1 lb. Represent it graphically. If the work of stretching the string is lost, what is the efficiency of this method of lifting?

(I.C.E., Feb., 1903.)

8. If B be the B.H.P. of a certain gas engine and I the corresponding I.H.P., and the results of two tests gave B = 57 corresponding to I = 73 and B = 117 corresponding to I = 139, what would the B.H.P. be if the I.H.P. were 35? Assume the law $B = xI + y$ where x and y are constants. (I.C.E., Feb., 1903.) (See my "*Elementary Applied Mechanics*," and use squared paper.)

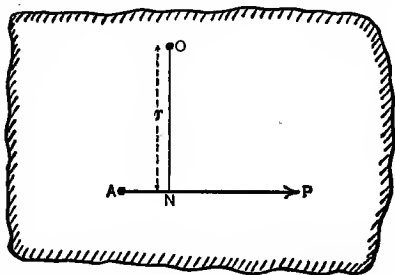
N.B.—See Appendices B and C for other questions and answers.

LECTURE III.

CONTENTS.—Moment of a Force—Definition of the Moment of a Force—Conventional Signs of Moments—Algebraic Sum of Moments—Equilibrium of a Body under the Action of Several Forces—Principle of Moments—Example I.—Couples—Definitions relating to Couples—Propositions I., II., and III.—Example II.—Work Done by Turning Efforts and Couples—Diagram of Work Done by a Couple of Uniform Moment—Work Done by Variable Moments—The Fusee—Correct Form to be given to the Fusee—Questions.

Moment of a Force.—When a body is free to turn about an axis, and is acted on by a force, P , whose line of action is in a plane perpendicular to the axis (but not passing through the same) the effect of P is to rotate the body about that axis.

The *measure* of this turning effect depends on two things, viz.—(1) *The magnitude of the force*, and (2) *The perpendicular distance between the axis and the line of action of the force*. Thus, if the axis be perpendicular to the plane of the paper, and O its intersection therewith, then the turning effect of P is measured by the product, $P \times ON$; ON being the length of the perpendicular from O upon the line of action, AP , of the force, P . This product is called the **Moment of the Force, P , with respect to the axis through O .**



MOMENT OF A FORCE.

When the force acts in a plane perpendicular to the axis, then it is best to define the moment of the force with respect to the point O ; the point O being the intersection of the axis with the plane of the force. We then get the following definition:—

The **Moment of a Force**, with respect to a point, is measured by the product of the force into the length of the perpendicular drawn from the given point to the line of action of the force.

From the above it will be seen that a force has no moment about a point in its own line of action.

In general, when we wish to find the moment of a given force with respect to a given *axis* in the body on which the force acts, we have to resolve the given force into two components, viz.—(1) One in a *plane* perpendicular to the axis; (2) The other perpendicular to this plane—i.e., parallel to the axis. The product of the former component into the length of the perpendicular from the axis upon its line of action, gives the required moment. The component parallel to the axis measures the thrust or pull *along* the axis. At the same time the component in the perpendicular plane gives a measure of the transverse pressure at the axis. The proof of these statements will be given immediately.

Conventional Signs of Moments—Algebraic Sum of Moments.—In problems relating to the moments of a number of forces acting on a body which is free to turn about a given axis, it is necessary to distinguish in sign between the moments of those forces which tend to turn the body in one direction about the axis, and those tending to turn the body in the opposite direction. If the moments of the one set of forces be regarded as *positive*, then those of the other set must be regarded as *negative*. Which direction of rotation is to be considered as the positive direction is a matter of little importance, so long as a distinction in sign is made and adhered to throughout the investigation.

By the term "*Algebraic Sum*" is to be understood the sum of the several quantities considered (whether moments or any other quantities differing in sign), each taken with its proper sign attached (+ or -).

Equilibrium of a Body under the Action of several Turning Forces.—The tendency of a force to turn a body about a given point depends only on the product of the two factors (1) effort and (2) its perpendicular distance from the point. It therefore follows that if any number of forces act in the same plane on a body and tend to turn it about a given point, the result will be the same (so far as the turning effect is concerned) as that of a single force acting in the same plane, and having a moment equal to the sum of the several moments. If some of the forces tend to turn the body in one direction and the others in the opposite direction; and, further, if the sum of the moments of the one set be equal to the sum of the moments of the other set, so that the *algebraical* sum of the moments is zero, it follows that the body will have no tendency to turn in the one direction more than in the other. In other words, the body will be in equilibrium so far as rotation is concerned.*

* The proofs of these statements are given in books on Theoretical Mechanics.

Principle of Moments.—If any number of forces, acting in the same plane, keep a body in equilibrium, then the sum of the moments of the forces tending to turn the body about any axis in one direction, is equal to the sum of the moments of the forces tending to turn the body about the same axis in the opposite direction.

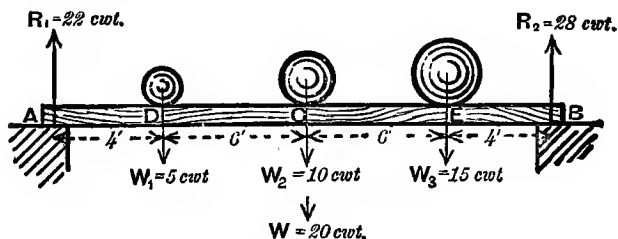
Conversely.—If the sum of the moments of the forces in the one direction is equal to the sum of the moments in the opposite direction, the body will be kept in equilibrium.

The *Principle of Moments* is sometimes stated in the following brief but useful form :—

When a body is kept in equilibrium by any number of co-planar forces, the algebraical sum of the moments of all the forces about any point in their plane is zero.

Conversely.—If the algebraical sum of the moments about any point in their plane is zero, the forces are in equilibrium.

EXAMPLE I.—A uniform beam weighing 1 ton rests on supports at its ends, 20 ft. apart. Weights of 5, 10, and 15 cwts. rest on the beam at distances of 6 ft. apart, the weight of 5 cwts. being 4 ft. from one of the supports. Find the reactions at the points of support.



TO ILLUSTRATE EXAMPLE ON MOMENTS.

ANSWER.—According to the *Principle of Moments* just stated, we may take moments about any point in the plane of the forces, in order to find R_1 and R_2 the reactions at the points of support. The student, however, will find it advantageous to take moments about *one* of the points of support; for then, the moment of the reaction at that point will vanish, and he will thus have an equation containing only one unknown quantity—viz., the other reaction.

Suppose we take moments about the point B, then we get

$$R_1 \times AB = W_1 \times DB + (W_2 + W) CB + W_3 \times EB$$

Substituting values, we get:—

$$R_1 \times 20 = 5 \times 16 + (10 + 20) \times 10 + 15 \times 4 = 440 \text{ (cwt.)}$$

$$\therefore R_1 = \frac{440}{20} = 22 \text{ cwts.}$$

Now, we can either take moments about A, and find R_2 in the same way as we have found R_1 ; or, we may make use of our knowledge of parallel forces (since the above system is one of parallel forces) and get R_2 . The latter method is the simpler. Adopting this method, we get:—

$$R_1 + R_2 = W_1 + W_2 + W + W_3$$

$$\therefore R_2 = 5 + 10 + 20 + 15 - 22 \text{ (cwts.)}$$

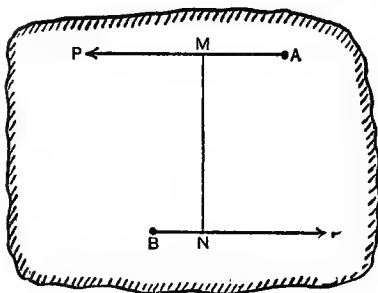
$$\text{Or, } R_2 = 28 \text{ cwts.}$$

Couples.—We shall now show that all questions relating to turning forces are really questions involving couples compounded with single forces.

DEFINITION.—A Couple is a system of two equal and oppositely directed parallel forces, whose lines of action do not coincide.

DEFINITION.—The Arm of a couple is the perpendicular distance between the two equal forces.

DEFINITION.—The Moment of a couple is the product of one of the equal forces into the arm.



MOMENT OF A COUPLE.

Thus, if a body be acted on by two equal and oppositely directed parallel forces, whose points of application are A and B respectively, then these forces constitute a Couple. If MN be drawn \perp to AP and then the length of this perpendicular is called the Arm of the Couple, and the product of the force P into the Arm MN is called the Moment of the Couple, and is denoted by $P \times MN$.

From an inspection of the figure it will be seen that the effect of a couple acting on a body is to produce rotation. A couple

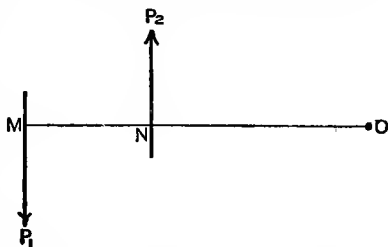
has no effect in producing translation of the body on which it acts.

We shall now prove the following important *Propositions* regarding couples.

PROPOSITION I.—The Algebraic Sum of the Moments of the two forces of a couple about any point in their plane is constant; or in other words,

The Moment of a Couple about any point in its plane is constant.

Let P_1 , P_2 be the equal forces constituting the couple and O be any point in the plane of the couple.



MOMENT OF A COUPLE ABOUT A POINT.

From O , drop the perpendicular ONM on the lines of action of P_1 and P_2 .

Then, Moment of P_1 about $O = P_1 \times OM$.

And, " P_2 " " " $= - P_2 \times ON$.

\therefore Moment of Couple about $O = P_1 \times OM - P_2 \times ON$.

i.e., " " " " $= P_1 \times MN$.

But $P_1 \times MN$ is clearly a constant quantity. It is, in fact, what we have already defined as the *Moment of the Couple*. Hence, we see that the moment of a couple about any point in its plane, is independent of the position of that point with respect to the couple.

Remembering, then, that a couple has no translatory effect on the body on which it acts, and that its rotatory effect is measured by its moment, we at once obtain the following corollaries from the above Proposition :—

(1) A Couple may be considered as acting anywhere in its own plane.

(2) A Couple may be replaced by another of equal moment and sign and acting in the same plane.

(3) The Resultant of two or more Couples acting in the same plane, is a couple whose moment is equal to the algebraic sum of the moments of the component couples.*

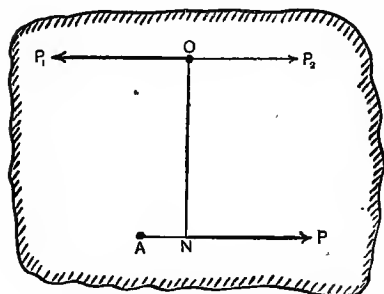
* Independent proofs of these propositions are usually given in books on Theoretical Mechanics.

PROPOSITION II.—A force acting on a rigid body can always be replaced by an equal force acting at any given point together with a couple.

Let P be a force acting at the point, A , in a rigid body, and let O be the given point.

At O , introduce two opposite forces, P_1 and P_2 , each equal to P , and having their line of action, $P_1 O P_2$, parallel to $A P$.

Then, obviously, the introduction of these two equal and opposite forces at O will not affect the action of P at A . We have now a system of three forces acting on the body, which is



A FORCE REPLACED BY A FORCE
AND A COUPLE.

equivalent to the single force, P , at A . But, clearly, two forces of this system—viz., P and P_1 —constitute a *couple*, the moment of which is $P \times ON$. The action of this couple is simply to produce rotation of the body. The remaining force, P_2 , is that part of the system which produces or tends to produce translation. The magnitude and direction of P_2 are always equal and

parallel, respectively, to those of the original force, P .

If O represents the intersection of the plane of the forces with an axis round which the body is free to turn, then the moment = $P \times ON$, and the transverse pressure on the axis is $P_2 = P$.

PROPOSITION III.—A force and a couple acting in the same plane are equivalent to or, may be replaced by, a single force in that plane.

This is the converse of *Proposition II.*, and might have been assumed here without proof; but we prefer giving a proof since it exhibits a method or process of reasoning useful for other purposes.

Let a force, P , and a couple whose moment is $Q \times q$, act in the same plane on a rigid body.

Replace this couple by another of equal moment and similar in sense, and having its forces each equal to P , the given force. [See Cors. (1) and (2), Prop. I.]

Let the arm of this new couple be p . Then we must have:—

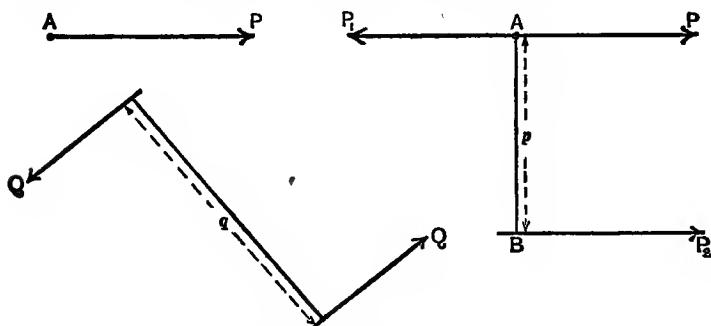
$$P \times p = Q \times q.$$

Or,

$$p = \frac{Q \times q}{P}.$$

Now, rotate this new couple in its plane into such a position that one of its forces acts along the same line as the given force P at A , but having its direction opposite to that of P . [See Cor. (1) Prop. I., and right hand fig. below]. We have then a system of three equal forces, two of which—viz., P, P_1 at A —neutralise each other, and then we are left with the single force P_2 , acting along a line BP_2 parallel to AP , and at a distance p from it, such that $p = qQ/P$.

Several important applications of the preceding propositions will be met with throughout the present treatise.



A FORCE AND A COUPLE REPLACED BY A FORCE.

EXAMPLE II.—A uniform platform, AC , turning about a hinge at A , is kept in a horizontal position by means of a chain, CH , fixed to a hook, H , in the wall vertically over A . A barrel weighing 6 cwts. is placed on the platform at B . Determine the tension in the chain, and the magnitude and direction of the reaction at the hinge, A ; given weight of platform = 2 cwts., $AC = 6$ feet, $AB = 5$ feet, and $AH = 8$ feet.

ANSWER.—(1) To find T the tension in the chain, CH .

From A drop the perpendicular AN on CH . Take moments about the hinge, A , so as to eliminate the reaction at that point. Then, by the *Principle of Moments*, we get:—

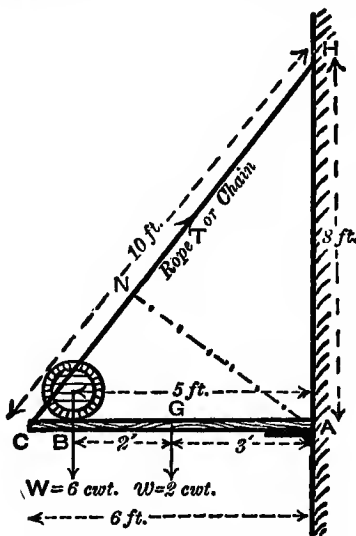
$$T \times AN = W \times AB + w \times AG$$

$$,, \quad ,, = 6 \times 5 + 2 \times 3 = 36 \text{ cwt.-ft.}$$

$$\therefore T = \frac{36}{AN} \text{ cwt.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We have now to determine the length of AN in feet.

The easiest way to find this, is to express the area of the right angled triangle, $A C H$, in two ways, and then equate these.



TO ILLUSTRATE THE TENSION IN THE CHAIN AND THE REACTION AT THE PLATFORM HINGE.

Thus, Area $\triangle A C H = \frac{1}{2} (C H \times A N)$.

Also, " " = $\frac{1}{2} (A C \times A H)$.

$\therefore \frac{1}{2} (C H \times A N) = \frac{1}{2} (A C \times A H)$.

i.e., $A N = \frac{A C \times A H}{C H}$.

Or, " = $\frac{6 \times 8}{10} = \frac{48}{10}$ ft., (2)

But, $C H = \sqrt{A C^2 + A H^2}$.

Or, " = $\sqrt{6^2 + 8^2} = 10$ ft.

From eqn. (2) $A N = \frac{48}{10} = 4.8$ ft.

And, " (1) $T = \frac{36}{4.8} = 7.5$ cwts.

(2) To find the reaction at the hinge, A.

Resolve the tension, T , in the chain, CH , into two components, viz., one along CA and the other perpendicular to CA .

Let T_h = horizontal component of T

„ T_v = vertical „ „

Then, $T_h = T \cos \angle H O A$

$$\therefore \quad \text{,,} = 7.5 \times \frac{6}{10} = 4.5 \text{ cwts.}$$

Also, $T_{\parallel} = T \sin \angle HCA$

$$\therefore \quad \text{,,} = 7.5 \times \frac{8}{10} = 6 \text{ cwts.}$$

Now, let R denote the reaction of the hinge at A , and let R_h, R_v represent the horizontal and vertical components of R . Then since the only horizontal forces acting on the platform are R_h and T_h , these must be equal and act in opposite directions.

$$\therefore R_h = T_h = 4.5 \text{ cwts.}$$

Again, R_v , T_v , W , and w constitute a system of parallel forces in equilibrium.

$$\therefore R_p + T_p = W + w$$

$$\therefore R_y = 6 + 2 - 6 = 2 \text{ cwts.}$$

But, $R^2 = R_h^2 + R_v^2$

$$\text{i.e.,} \quad R^2 = 4.5^2 + 2^2 = 24.25$$

$$\therefore R = \sqrt{24.25} = 4.92 \text{ cwts.}$$

(3) To find the direction of the reaction, R.

Since R_h acts from A to C, and R_v acts vertically upwards, it at once follows that the direction of R lies along some line between AC and AH.

Let r denote the length of the perpendicular from C upon the line of action of R. Then, taking moments about C, we get by the *Principle of Moments* :—

$$\mathbf{R} \times r = \mathbf{W} \times \mathbf{CB} + w \times \mathbf{OG}$$

Or, $4.92 \times r = 6 \times 1 + 2 \times 3 = 12 \text{ cwt.-ft.}$

$$\therefore r = \frac{12}{4.92} = 2.43 \text{ ft.}$$

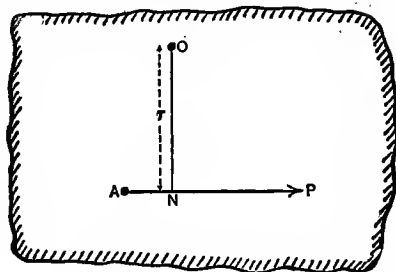
To set out the line of action of R we may proceed thus:—With centre, C , and radius, $r = 2.43$ ft. draw an arc of a circle above line AO . From A draw a tangent to this arc. Then the direction of the reaction is along this tangent.

Note.—When the student has studied the Lectures on the Graphical Methods of determining the stresses in structures he should return to this problem, and determine graphically the tension in the chain, CH , and the magnitude and direction of the reaction of the hinge at A .

Work Done by Turning Efforts and Couples.—We are now in a position, to be able to find the work done by turning forces and couples, and to construct the diagrams of work done by such efforts.

CASE I.—Work Done by Uniform Moments.—Let O be a point in the plane of the turning or twisting effort, P , round which the body is rotating.

Then, Moment of P about $O = M = P \times ON = Pr$.



UNIFORM MOMENT.

Let the body make n complete turns.

Then, Displacement of $P = 2\pi rn$

\therefore Work done by $P = P \times 2\pi rn = Pr \times 2\pi n = M \times 2\pi n$.

But $2\pi n$ is the circular measure of the angle turned through by the effort, P .

Let θ denote this angle—i.e., let $\theta = 2\pi n$,

Then, Work done by moment, M , during angular displacement, θ } $= M\theta$, . . . (1)

Now we have already seen that every turning effort may be regarded as equivalent to a couple of equal moment to the effort and a force equal and parallel to the effort acting at the point O .

For the present we are only concerned with the turning effect of the effort or couple; hence, if a body is rotated by a couple of moment, M , through an angular displacement, θ :—

Then, Work done by couple = $M \theta$, (2)

Where, $\theta = 2 \pi n$.

Diagram of Work Done by a Couple of Uniform Moment.—From equations (1) and (2) it will be seen, that the quantity, θ , or $2 \pi n$, has the same relation to the equation for the work done by a couple, that L had in the previous expressions ($P \times L$) for the work done by a force; only, that here θ represents an *angular* displacement while in the previous case L represented a *linear* displacement. And, just as we can construct a diagram of work for linear displacements, so also, can we construct a similar diagram of work for angular displacements.

Hence, draw two rectangular axes, ox , oy . Set off OA to represent the turning moment, M , and OB to represent the angular displacement, θ .

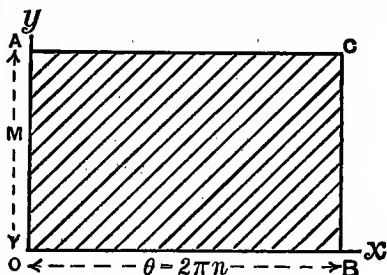


DIAGRAM OF WORK FOR A
UNIFORM MOMENT.

Then, if the moment of the couple be uniform, the area of the rectangle $OACB$ represents the work done.

i.e., Work Done = Area $OACB$ = $M \theta$.

CASE II.—Work Done by a Couple of Variable Moment.—If we wind up a flat spring (such as the main spring of a watch or clock) or twist a helical spring or a wire or shaft by an effort in a plane perpendicular to its axis, the twisting moment required is proportional to the angle of twist within certain limits. This law may be stated briefly, thus:—

$$M \propto \theta.$$

We can prove, as in Lecture I., that the diagram of work for such cases as the above will be a triangle or a trapezoid according as the spring or shaft is in a neutral or initial state of stress when we begin to further twist or untwist it.

Thus, let the material be in an unstressed condition to begin with; and let M_2 be the twisting moment corresponding to the

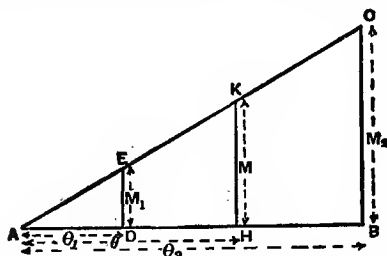


DIAGRAM OF WORK FOR A VARIABLE MOMENT.

angle of twist, θ_2 . Then, for any other angle of twist, θ , we get the corresponding twisting moment, M , from the proportion

$$M : M_2 = \theta : \theta_2.$$

Hence, if AB , AH , and BC represent θ_2 , θ , and M_2 respectively, we see that M will be represented by the ordinate HK .

For, obviously,

$$HK : BC = AH : AB = \theta : \theta_2.$$

\therefore HK represents the twisting moment for angle of twist, θ , to the same scale that BC represent M_2 .

$$\text{Hence, } \left. \begin{array}{l} \text{Work done in twisting} \\ \text{material through angle, } \theta \end{array} \right\} = \text{Area } \triangle AHK = \frac{1}{2} M \theta.$$

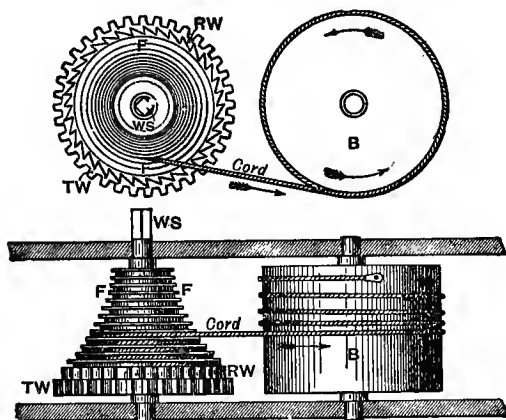
If the initial and final angles of twist be θ_1 and θ_2 respectively, then

$$\left. \begin{array}{l} \text{Work done in twisting material} \\ \text{through angle } (\theta_2 - \theta_1) \end{array} \right\} = \text{Area } DBCE$$

$$\begin{array}{ccc} \text{,,} & \text{,,} & = \frac{1}{2} (M_2 + M_1) (\theta_2 - \theta_1). \end{array}$$

The Fusee.—As an illustration of the manner in which the variable twisting moment of a coiled spring may be compensated, and thus secure a uniform turning effort, we may instance the case of the *fusee* as adopted in many watches, clocks, and chronometers. In such cases, the driving of the works at a constant rate is the object aimed at, and this naturally requires a constant turning effort in the wheelwork, this effort being just sufficient to overcome the frictional and other resistances offered by the mechanism. Now, one of the most compact and con-

venient pieces of mechanism into which mechanical energy can be stored is that of a coiled spring, and since the very nature of the spring is such, that its moment decreases as it uncoils, we must employ some compensating device between this variable driving force and the constant resistance. The fusee does this in a most accurate and complete manner. Looking at the accompanying figures and index to parts, we see that the barrel, B, which con-



THE FUSEE FOR A CLOCK OR WATCH.

INDEX TO PARTS.

B represents Barrel.
F ,, Fusee.
RW ,, Ratchet wheel.

TW represents Toothed wheel.
WS ,, Winding square.

tains the watch or clock spring, is of uniform diameter, and that between the outside of this barrel and the fusee, or spirally grooved cone, there passes a cord or chain. When the winding key is applied to the winding square, WS, and turned in the proper direction, a tension is applied to the cord, and it is wound upon the spiral cone, thus coiling up the spring inside the barrel, B; for the outer end of this spring is fixed to the periphery of the barrel, and the inner end to its spindle or axle. When the spring is fully wound up it exerts the greatest force, but it acts at the least leverage, since the cord is on the groove of least diameter. When the spring is almost uncoiled it acts at the greatest leverage, for then the cord is on the groove of largest diameter. Consequently, the radii of the grooves

of this cone are made to increase in proportion as the force applied to the cord decreases, in order that there shall be a constant turning effort on the works of the clock or watch.

Correct Form to be given to the Fusee Curve.—We shall now show that the true form of the fusee curve is that of a rectangular hyperbola for equalising the effect of a spring of uniform elasticity, and when neglecting the other connections.

Let $ABCD$ be the diagram of work for the spring inside the barrel, B . Then, from what has been said above, $ABCD$ will be a trapezoid.

Let BC represent P , the force which the spring (inside the barrel) exerts on the cord or chain when it is fully wound up.

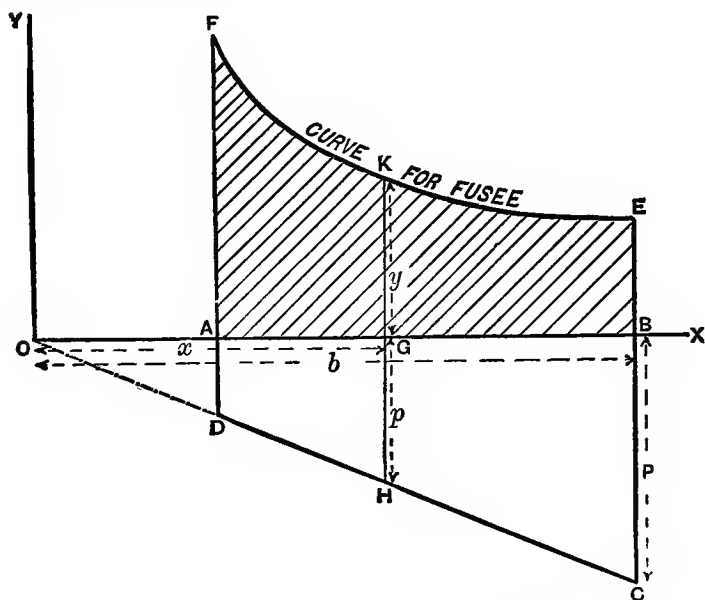


DIAGRAM OF WORK, &c., FOR THE FUSEE.

Similarly, let GH represent p , the tension in the cord or chain at any other instant.

Let the ordinates EB , KG , and FA , represent the several radii at which the cord acts on the fusee. Then EKF will be the curve required to be given to the fusee. Thus, BE represents the radius at which the tension, P , in the cord acts when the spring

inside B is completely wound up. Similarly, G K represents the radius of the fusee corresponding to the tension, p , in the cord.

Produce C D till it meets B A at O. Take O as the origin. Let O G = x , G K = y , and O B = b .

Then, if we have a *constant twisting moment* acting on the fusee spindle :—

$$p \times y = \text{a constant } (m) \quad . \quad . \quad . \quad (1)$$

But, $p : P = x : b$

$$\therefore p = \frac{P}{b} x = n x$$

Where $n = \frac{P}{b} = \text{a constant.}$

Substituting this value for p in equation (1) we get :—

$$n x y = m$$

Or, $x y = \frac{m}{n} = \text{a constant.}$

But this is the equation for a rectangular hyperbola referred to the axes O X, O Y which are its asymptotes. We have met with this curve before when treating of the expansion or compression of gases according to Boyle's law.

In practice, the fusee is made as nearly as possible to this shape. Then the fusee and spring are connected, as shown by the figures on page 43, and tested by fixing an L-shaped lever (with an adjustable weight on the long arm of this adjusting rod) to the winding square, W S, and finding whether the tension in the cord or chain (as due to the spring enclosed in B) is balanced in every position by the same turning effort on the lever.

Should the turning moment of the combined spring and fusee be thus found to be greater when the spring is fully wound up than when it is nearly run down, the initial tension of the spring is too great. To lessen this, the ratchet wheel, R W, is eased back a tooth or two, the cord readjusted, and the above experiment repeated until the nearest approach is arrived at to a uniform turning effort on the works of the timepiece.

LECTURE III — QUESTIONS.

1. Define the moment of a force with respect to a point. When is a moment reckoned positive and when negative? Draw an equilateral triangle, ABC , and suppose each side to be 4 feet long. A force of 8 units acts from A to B , and a force of 10 units from C to A . (a) Find the moment of each force with respect to the middle point of BC ; (b) Find a point with respect to which the forces have equal moments of opposite signs. *Ans.* (a) $18\sqrt{3}$; (b) Any point on the resultant.

2. State the principle of moments and hence show that the moments of two forces about any point in their resultant are equal and opposite. A rod is supported horizontally on two points, A and B , 12 feet apart. Between A and B points C and D are taken such that $AC = BD = 3$ feet. A weight of 120 lbs. is hung at C , and a weight of 240 lbs. at D . Take a point O midway between A and B and find with respect to O the algebraic sum of the moments of the forces acting on the rod on one side of O . (You may neglect the weight of the rod.) *Ans.* 540 lbs.-ft.

3. In a blowing engine of the overhead beam construction the area of the steam piston is 2,712 square inches, and the mean pressure of the steam is 30 lbs., while the area of the piston of the blowing cylinder is 16,272 square inches. The leverage of the working beam is as 15 on the steam side to 20 on the opposite side; what is the pressure of the air as it leaves the blowing cylinder? *Ans.* 3.75 lbs. per square inch.

4. A safety valve, 3 inches in diameter, is held down by a lever and weight. The distance from the fulcrum to the pin of the valve is 6 inches. Weight of valve 5 lbs. Weight of lever 15 lbs. Distance from fulcrum to centre of gravity of lever 16 inches. Find where a weight of 60 lbs. must be placed on the lever so that the steam may blow off at a pressure of 56 lbs. per square inch. *Ans.* 35.1 inches from fulcrum.

5. Define a couple, its arm, and its moment. Show that two couples, whose moments are equal and of opposite signs, are in equilibrium when they act in the same plane on a rigid body. If forces act from A to B , B to C , and C to A , along the sides of a triangle, ABC , and are proportional to the sides along which they respectively act, show that they are equivalent to a couple.

6. Show that a force acting at a given point A , may be replaced by an equal parallel force acting at any other point B , and a couple whose moment equals moment of original force about B .

7. Find the resultant of a force and a couple acting in the same plane. Draw a square, $ABCD$, and its diagonal, AC . Two forces of 10 lbs. each act from A to B and from C to D respectively, forming a couple. A third force of 15 lbs. acts from C to A . Find their resultant, and show in a diagram exactly how it acts. *Ans.* $R = 15$ lbs.

8. State the principle of moments, and apply it to the solution of the following question:— AB , AC are sheer poles secured to a base plate in the ground at B and C , and held in position by a wire guy or tension rope, AE , attached to the ground at E , D is the middle point of the line joining B and C , and BC is perpendicular to ED . Given $AB = AC = 25$ feet; $BC = 14$ feet; $DE = 40$ feet; $AE = 55$ feet. Find tension in the guy rope when a weight of 20 tons is suspended from A . *Ans.* 13.55 tons.

9. Explain, with a sketch, the use of a fuses in equalising the variable force of a spring coiled within the barrel of a watch. Find the theoretical form to be given to the curve of the fusee.

10. A safety valve, $3\frac{1}{2}$ inches in diameter, is held down by a lever and spring. The arrangement has to be so constructed that each pound of additional pressure per square inch on the valve will be registered as such on the spring at the end of the lever. Neglecting the weights of the lever and valve, you are to determine the relative distances of spring and valve from the fulcrum of the lever. After the valve has been set, determine the additional pressure per square inch which will be necessary to lift the valve $\frac{1}{16}$ inch, the spring requiring a force of 10 lbs. to extend it 1 inch. You may neglect the weights of the lever, valve, and spring. Sketch the arrangement. *Ans.* 9.625:1; 4.8 lbs.

11. A lever safety valve has the following dimensions:—Mean area of valve 3.2 square inches; weight of valve and spindle 9 lbs.; horizontal distance from centre of valve to fulcrum of horizontal lever 3.4 inches; weight of lever 13 lbs.; distance of centre of gravity of lever from fulcrum 8 inches; the movable weight is 67 lbs. What is the distance along the lever through which the weight ought to be shifted for a difference of pressure of 10 lbs. per square inch? *Ans.* 1.62 inches. (B. of E., Adv., 1900.)

N.B.—See *Appendices B and C* for other questions and answers.

LECTURE III.—A.M. INST. C.E. EXAM. QUESTIONS.

1. Define the moment of a force about a point, and show that the sum of the moments of any two forces about a point in their plane is equal to the moment of their resultant about the same point. (I.C.E., Oct., 1897.)

2. Given a system of forces whose lines of action meet in a point, how would you find the resultant? Show briefly that any system of forces is reducible to a single force, together with a single couple. (I.C.E., Feb., 1898.)

3. A weight of 1,000 lbs. is carried by three convergent ropes 10, 10, and 15 feet long, hanging respectively from points A, B, C, in a horizontal ceiling, such that AB is 16 feet, and BC and CA are each 20 feet. Find the pull on each rope. (I.C.E., Feb., 1898.)

4. A platform in the shape of an equilateral triangle, with sides 6 feet long, supported at its three corners, carries a weight 3 feet from one corner and 4 feet from another. Find the fraction of the weight borne by each of the three supports. (I.C.E., Feb., 1898.)

5. Give a graphical method for determining the resultant of a number of forces in one plane—(i.) when their lines of action all pass through one point; (ii.) when they do not pass through one point. In the latter case is it always possible to find a single force which will replace the given forces? What part of the construction decides this? In such a case how could the system of forces be balanced? (I.C.E., Feb., 1899.) (See also my "*Elementary Applied Mechanics*.")

6. A vertical brick wall, 27 inches thick, stands 12 feet high above the ground. Assuming the brickwork to weigh 120 lbs. per cubic foot, calculate the uniform wind-pressure per square foot that would just suffice to overturn the wall upon one edge of its base—at ground level—neglecting any adhesion of the mortar. Also if the wall were subjected to the pressure of flood-water on one side, find the height to which the water must rise before overturning the wall. (I.C.E., Feb., 1899.)

7. What do you understand by the axis of a couple, and what particulars are necessary to fix its value. State the different ways in which a couple can be replaced by an equivalent one. (I.C.E., Oct., 1899.)

8. Prove that any system of forces acting on a rigid body can be reduced to a single force and a couple in a plane perpendicular to it.

(I.C.E., Oct., 1899.)

9. Show how the resultant of two couples can be found, if they act in (a) parallel planes, (b) planes not parallel. If four couples of 10, 18, 25, and 40 lbs.-feet act on a rigid body in planes which all intersect in a line but are inclined at angles of 30° , 60° , and 90° respectively with the plane of the first-mentioned couple, find their resultant. (I.C.E., Feb., 1900.)

10. Show how to find (graphically or otherwise) the pressures on the two supports of a horizontal beam which is loaded at any given points. If the distance between the supports be 20 feet, and if one of the loads be 12 cwts., find the changes in the pressure produced by shifting this load through a space of 5 feet along the beam. (I.C.E., Feb., 1901.)

11. The centre of the ball of a ball-tap is 15 inches from the fulcrum or pivot. The centre of the valve which closes the supply pipe (of $\frac{1}{2}$ inch diameter) is $1\frac{1}{2}$ inch from the fulcrum. The ball is 6 inches in diameter, and the valve is closed when the ball is half immersed. If the weight of the ball and lever acts as a force of 1 lb. at a distance of 9 inches from the fulcrum, find the greatest pressure of water in the pipe against which the ball will keep the valve closed. Sketch the arrangement. (I.C.E., Oct., 1901.)

12. A solid is made up of two cylinders, with their axes in the same straight line; one cylinder is 6 inches long and 3 inches diameter, and the

second is 8 inches long and 2 inches diameter. Find the centre of gravity of the solid. If the solid rests on a rough inclined plane so that it will not slide, the cylinder 3 inches in diameter being the lower one, find the angle of inclination of the plane to the horizontal at which the solid will just not topple over. (I.C.E., Oct., 1901.)

13. Prove that "the resultant of two like parallel forces is equal to their sum, parallel to their direction, and applied at a point found by dividing the line between them inversely as the forces," and also, that "the moment of a couple is constant about any point in its plane."

(I.C.E., Oct., 1901.)

14. State the conditions of equilibrium of three forces acting upon a body when the forces are not parallel and act in one plane. A trap-door is 4 feet square and weighs 75 lbs. It is hinged by one of its horizontal edges, and is supported with its plane horizontal by a chain which passes from the middle point of the outer edge of the door to a point vertically over the middle point of the edge in which the hinges are fixed, and 7 feet above it. Determine graphically the pull in the chain and the thrust on the hinges. (I.C.E., Feb., 1902.)

15. Under what conditions will two couples acting on a body exactly balance each other? The davits supporting a boat are fastened to a ship by passing through a vertical hole in the gunwale, and by resting with the lower end in a step on the deck. Show how to determine the stress on the gunwale. (I.C.E., Oct., 1902.)

N.B.—See *Appendices B and C* for other questions and answers.

LECTURE IV.

CONTENTS.—Principle of Work—Principle of the Conservation of Energy—Definition of Energy—Useful and Lost Work in Machinery—Proposition—Principle of Work Applied to Machines—Definition of Efficiency—Object of a Machine—Definition of a Machine—Simple or Elementary Machines—Force Ratio—Velocity Ratio—Mechanical Advantage—Relations between the Advantage, Velocity Ratio, and Efficiency of a Machine—Examples I. and II.—Questions.

BEFORE taking up the subject of simple machines we shall give a brief statement of another important principle in Mechanics known as the “principle of work.”

Principle of Work.—If a body or system of bodies be in equilibrium under the action of any number of forces, and receive a small displacement, the algebraical sum of the work done by all the forces is zero.

Conversely.—If the work done be zero, the forces are in equilibrium.

We may verify the truth of the principle of work by assuming the principle of moments, or the principle of the parallelogram of forces, &c.; or, conversely; having assumed the principle of work we can verify the truth of the principle of moments, or the principle of the parallelogram of forces. After all, the principle of work is only a particular case of the more general principle called the **Principle of the Conservation of Energy**, which is now universally accepted by all scientists, and may be stated thus:—

Principle of the Conservation of Energy.—The total energy of any material system is a quantity which can neither be increased nor diminished by any action between the parts of the system, though it may be transformed into any of the forms of which energy is susceptible. (*Clerk Maxwell.*)

DEFINITION.—Energy confers upon a body possessing it the ability to do work.

The principle of the conservation of energy, therefore, asserts that there can be no increase or decrease in the energy of any system without an equivalent loss or gain of energy in some other system. If in any isolated system there be an increase in one form of energy this can only happen at the expense of some of the other forms of energy in the system.

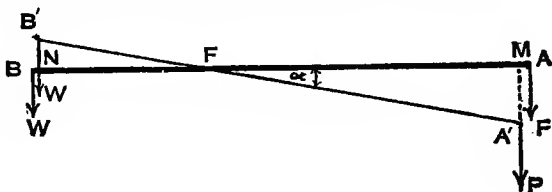
Hence, the total energy in the Universe is a constant quantity.

We may change energy of one form into that of another, but we can never change the total amount.

Useful and Lost Work in Machinery.—In the case of machines, it is true, that the useful work is much less than the work put into the machine. A part of the total energy exerted is rendered unavailable for useful work, this part being employed in overcoming the friction at the rubbing surfaces; by setting up vibrations in the machinery, &c.; and these reappear in the forms of heat and sound energy, &c. The energy thus mis-spent is a direct loss to the engineer, and he has to contrive means for its reduction; although, he can never hope to entirely eliminate it.

Since the student is still expected to give a demonstration or verification of the *Principle of Work*; when, say, the truth of the *Principle of Moments* is assumed, we herewith give such a demonstration in its usual form as it appears in most works on this subject.

PROPOSITION.—To verify the truth of the Principle of Work by assuming the truth of the Principle of Moments.



PRINCIPLE OF WORK AND OF MOMENTS.

Let AFB be a rigid lever capable of turning about a fulcrum at F. Let forces P and W act at the extremities A and B respectively. Let the three forces, P, W, and the reaction at F, be a system of forces in equilibrium.

Then, by the *Principle of Moments*, we have :—

$$P \times AF = W \times BF$$

$$\text{Or,} \quad P : W = BF : AF \quad (1)$$

Now, conceive the system to receive a small displacement, the forces being still in equilibrium. For this displacement it is best to conceive the lever tilted through a very small angle α , round the fulcrum, F; its new position being A'FB'.

Then,

$$\text{Work done by } P = P \times A' M$$

$$,, \quad W = - (W \times B' N)$$

$$\therefore \quad \text{Total work done} = P \times A' M - W \times B' N \quad (2)$$

But, from the similar Δ^s , $N F B'$ and $M F A'$, we get:—

$$B' N : A' M = B' F : A' F$$

$$,, \quad = B F : A F$$

$$\therefore \text{ From eqn. (1), } B' N : A' M = P : W$$

$$\therefore \quad W \times B' N = P \times A' M$$

Hence, from eqn. (2), we get:—

$$\text{Total work done} = P \times A' M - P \times A' M = 0.$$

This verifies the principle as stated above.

The student should now prove in a similar manner the converse statement, and also, verify the truth of the *Principle of Moments* by assuming the *Principle of Work*.

Principle of Work Applied to Machines.—When applied to machines the *Principle of Work* takes the form:—

Total work expended = Useful work done + Work lost in the machine.

Or, Work put in = Work got out + Lost work.

If we denote these three quantities by W_T , W_U , and W_L respectively, we can write the above equation thus:—

$$W_T = W_U + W_L \quad (I)$$

DEFINITION.—The ratio which the useful work done bears to the total work expended is called the efficiency of the machine.

$$\text{i.e.,} \quad \text{Efficiency} = \frac{\text{Useful work done}}{\text{Total work expended}} = \frac{W_U}{W_T} \quad (II)$$

The efficiency of an actual machine is always a proper fraction. The efficiency could only be unity in the case of a perfect machine, or where we assume the entire absence of frictional and other losses. In such theoretical cases we state the *Principle of Work* in the following form:—

$$\text{Total work expended} = \text{Useful work done.}$$

Object of a Machine.—The object of a machine is to enable us to perform work of various kinds, either by our muscular exertions or by utilising the forces of nature.

We may define a machine either from a *statical* or from a

kinematical point of view. Regarded Statically, it is an instrument for changing the magnitude, direction, or place of application of a given force. Kinematically, it is an instrument for changing the direction or the velocity of a given motion, or both direction and velocity.

Combining these two statements, we get the following :—

DEFINITION.—A machine is an instrument, or combination of movable parts, constructed for the purpose of transmitting and modifying, in various ways, force or motion, or both force and motion.

Or, a machine may be defined to be a combination of resistant bodies whose relative motions are completely constrained, and by means of which the natural energies at our disposal may be transformed into any special form of work. (*Prof. A. B. W. Kennedy.*)

Simple or Elementary Machines.—All machines, however complicated, are merely combinations of two or more of the following mechanisms :—

- | | |
|---------------------------|------------------------|
| 1. The Lever and Fulcrum. | 4. The Inclined Plane. |
| 2. The Pulley. | 5. The Wedge. |
| 3. The Wheel and Axle. | 6. The Screw. |

In reality, there are only *two* elementary mechanisms distinct in principle—viz., the Lever and the Inclined Plane. The Pulley and the Wheel and Axle are but modifications of the Lever; whilst the Wedge and the Screw are but particular cases of the Inclined Plane.*

Force Ratio—Velocity Ratio—Mechanical Advantage.—In considering any machine it is desirable to know the ratio which the applied force or effort bears to the resistance or load overcome. This is termed the **Force Ratio**. Also, the ratio of the velocities of the points of application of the effort and resistance. This is termed the **Velocity Ratio**. In this treatise we shall denote the applied force or effort by P or Q according as frictional resistance is neglected or taken into account; W being the resistance or load in both cases.

Then, P = *Theoretical* force required to overcome resistance, W.

And, Q = *Actual* " " " " W.

$$\begin{array}{lcl} \text{Also,} & \text{Theoretical Force Ratio} & = \frac{P}{W} \\ \text{And,} & \text{Actual Force Ratio} & = \frac{Q}{W} \end{array} \left. \begin{array}{l} \\ \\ \end{array} \right\} \dots \dots \dots \text{(III)}$$

* For descriptions, &c., of these simple mechanisms, see the author's *Elementary Manual of Applied Mechanics*.

Let V = Velocity of the point of application of the effort P or Q .

" v = " " " " " resistance, W .

Then, Velocity Ratio = $\frac{\text{Velocity of } P \text{ or } Q}{\text{Velocity of } W} = \frac{V}{v}$. . . (IV)

Let x = Displacement of the point of application of P or Q .

" y = " " " " " W .

Then, in a given time or period of motion of the machine, it is clear that :—

$$\frac{\text{Velocity of } P \text{ or } Q}{\text{Velocity of } W} = \frac{\text{Displacement of } P \text{ or } Q \text{ in a given time}}{\text{Displacement of } W \text{ in the same time}}.$$

$$\text{Or, } \left. \begin{aligned} \frac{V}{v} &= \frac{x}{y} \text{} \\ \text{i.e., Velocity Ratio} &= \frac{x}{y} = \frac{P \text{ or } Q's \text{ displacement}}{W's \text{ displacement}} \end{aligned} \right\} \text{ . . . (V)}$$

The reciprocal of the force ratio is usually spoken of as the *Mechanical Advantage* of the machine.* Hence :—

$$\text{Theoretical advantage} = \frac{\text{Resistance overcome}}{\text{Theoretical force required}} = \frac{W}{P} \quad \text{(VI)}$$

$$\text{Actual advantage . . .} = \frac{\text{Resistance overcome}}{\text{Actual force required}} \text{ . . .} = \frac{W}{Q} \quad \text{(VII)}$$

Relations between the Advantage, Velocity Ratio, and Efficiency of a Machine.—Neglecting friction and applying the "*Principle of Work*" to any machine, we get :—

$$W \times \text{its displacement} = P \times \text{its displacement.}$$

$$\text{Or, } W \times y = P \times x.$$

$$\therefore \left. \begin{aligned} \frac{W}{P} &= \frac{x}{y} \\ \text{Or, from Equation (V), } \frac{W}{P} &= \frac{V}{v} \end{aligned} \right\} \text{ (VIII)}$$

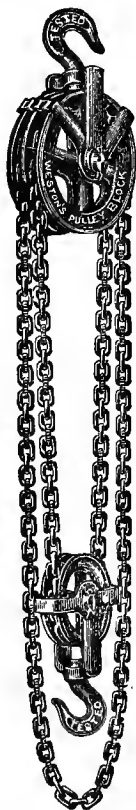
i.e., Theoretical Advantage = Velocity Ratio.

* In some treatises on Applied Mechanics the *force ratio* and *Mechanical Advantage* are synonymous terms, but, since in many problems it is desirable to know what ratio P or Q bears to W , we have chosen the former term (*force ratio*) to denote either of these ratios, retaining the term "*advantage*" in its original sense, to denote the *reciprocal* of the *force ratio* or the ratio of the load overcome to the force required to overcome it. The term *Hypothetical Mechanical Advantage* is sometimes used instead of *Theoretical Advantage*.

Again, in any machine we get:—

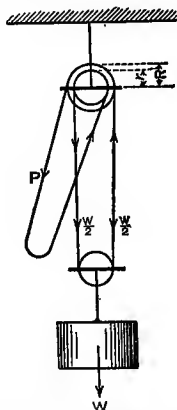
$$\begin{aligned} \text{Efficiency} &= \frac{\text{Useful work done}}{\text{Total work expended}} = \frac{W y}{Q x} \\ \text{,,} &= \frac{W}{Q} \times \frac{P}{W} \text{ from Equation (VIII).} \\ \text{,,} &= \frac{P}{Q} \dots \dots \dots \end{aligned} \left. \begin{array}{l} \\ \\ \end{array} \right\} \dots \text{(IX)}$$

i.e., $\text{Efficiency} = \frac{\text{Theoretical force to overcome } W}{\text{Actual force to overcome } W}$



WESTON'S DIFFERENTIAL PULLEY BLOCK.

EXAMPLE I. — Determine the relation between P and W in Weston's differential pulley block—(1) by the Principle of Moments, (2) by the Principle of Work. The radii of the pulleys are $4\frac{1}{2}$ inches and $4\frac{1}{4}$ inches. Taking the efficiency of the machine at 40 per cent., find the effort required on the



SKETCH OF WESTON'S DIFFERENTIAL PULLEY BLOCK.

required on the hauling chain in order to raise a weight of $\frac{1}{4}$ ton. What is the actual advantage in this machine?

ANSWER.—Let R and r denote the radii of the larger and smaller pulleys respectively. Then:—

(1) Since the weight, W , is supported by two parts of the chain, it is clear that the tension in each part is $W/2$. Considering the upper or differential pulley we see that it is acted on by three forces at the circumferences, viz.:—The tensions in the two parts of the chain supporting W , and the pull, P , along the hauling part of the chain.

Taking moments round the centre of the pulley pin, and applying the Principle of Moments, we get :—

$$P \times R + \frac{W}{2} \times r = \frac{W}{2} \times R.$$

$$\text{Or,} \quad P \times R = \frac{W}{2} (R - r),$$

$$\therefore \quad \frac{P}{W} = \frac{R - r}{2R}.$$

(2) Suppose the system to receive such a displacement that the differential pulley makes one complete turn, W being raised during the operation. Then :—

The displacement of $P = x = 2\pi R$.

One part of the chain supporting W is overhauled by an amount $= 2\pi R$, while the other part is let out by a length $= 2\pi r$. The weight, W , will, therefore, be raised by an amount equal to the *algebraical* mean of these two displacements of the supporting chain.

Or,

$$\text{Displacement of } W = y = \frac{1}{2} (2\pi R - 2\pi r) = \pi (R - r).$$

Hence, by the *Principle of Work*, we get :—

$$Px = Wy; \text{ or, } \frac{P}{W} = \frac{y}{x}.$$

$$\text{i.e.,} \quad \frac{P}{W} = \frac{\pi (R - r)}{2\pi R} = \frac{R - r}{2R}.$$

This is, however, the same result as before.

$$\text{Efficiency} = \frac{\text{Useful work done}}{\text{Total work expended}} = \frac{Wy}{Qx} = \frac{W}{Q} \times \frac{R - r}{2R}.$$

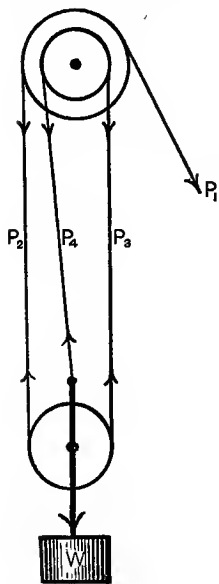
In the example, the efficiency = 40 per cent. = $\cdot 4$; $W = 560$ lbs.; $R = 4\frac{1}{2}$ "; $r = 4\frac{1}{4}$ ".

$$\text{Hence,} \quad \cdot 4 = \frac{560}{Q} \times \frac{4\frac{1}{2} - 4\frac{1}{4}}{2 \times 4\frac{1}{2}} = \frac{560}{Q} \times \frac{1}{36},$$

$$\therefore \quad Q = \frac{560}{\cdot 4 \times 36} = 38\cdot 8 \text{ lbs.}$$

$$\text{Actual advantage} = \frac{W}{Q} = \frac{560}{38\cdot 8} = \frac{14\cdot 4}{1}.$$

EXAMPLE 11.—It is found by trial that when P is on the point of lifting Q by means of a single fixed pulley, $P = (1 + m) Q$, where m is a fraction depending on the friction of the parts of the machine. If three such pulleys are combined into “a block and tackle,” find the effort requisite to raise a given weight by means of it. If m equals 0.2 so that an effort of 60 lbs. would just raise a weight of 50 lbs. in a single fixed pulley, find the number of ft.-lbs. of work done against friction, when a weight of 1,000 lbs. is raised 20 ft. by means of a block and tackle of three such pulleys. (S. and A. Adv. Theor. Mechs. Exam. 1883.)



QUESTION ON THE
PULLEY BLOCK.

ANSWER.—(1) Let the figure represent the block and tackle consisting of three pulleys or sheaves. (In this figure we have drawn the pulleys of different sizes for the purpose of showing clearly the various ropes and exhibiting the forces).

Let P_1 be the effort required to raise a given weight, W , by means of this system of blocks. Let P_2, P_3, P_4 denote the tensions in the three parts of the rope as shown.

Then, from conditions stated in question, we get :—

$$P_1 = (1 + m) P_2 \quad . \quad . \quad . \quad (1)$$

$$P_2 = (1 + m) P_3 \quad . \quad . \quad . \quad (2)$$

$$P_3 = (1 + m) P_4 \quad . \quad . \quad . \quad (3)$$

Adding together the corresponding sides of equations (1), (2), and (3), we get :—

$$P_1 + P_2 + P_3 = (1 + m) (P_2 + P_3 + P_4).$$

$$" \quad " \quad = P_2 + P_3 + P_4 + m (P_2 + P_3 + P_4).$$

$$i.e., \quad P_1 = P_4 + m (P_2 + P_3 + P_4).$$

$$\text{But,} \quad P_2 + P_3 + P_4 = W.$$

$$\therefore \quad P_1 = P_4 + m W \quad . \quad . \quad . \quad (4)$$

Multiplying together equations (1), (2), and (3), we get :—

$$P_1 P_2 P_3 = (1 + m)^3 P_2 P_3 P_4.$$

$$\therefore \quad P_1 = (1 + m)^3 P_4.$$

$$\text{Or,} \quad P_4 = \frac{P_1}{(1 + m)^3}$$

Substituting this value for P_1 in equation (4), we get:—

$$P_1 = \frac{P_1}{(1+m)^3} + m W.$$

$$\therefore P_1 \{ (1+m)^3 - 1 \} = m (1+m)^3 W.$$

$$\text{Or, } (m^3 + 3m^2 + 3m) P_1 = m (1+m)^3 W.$$

$$\text{i.e., } (m^2 + 3m + 3) P_1 = (1+m)^3 W.$$

$$\text{Or, } \left. \begin{aligned} P_1 &= \frac{(1+m)^3}{m^2 + 3m + 3} W \end{aligned} \right\} \quad (5)$$

(2) In the example given, $m = 0.2$, and we may find the effort, P_1 , required to raise a weight, $W = 1,000$ lbs.

From equation (5), we have:—

$$P_1 = \frac{(1.2)^3}{0.2^2 + 3 \times 0.2 + 3} \times 1,000 \text{ lbs.}$$

$$= \frac{1.728 \times 1,000}{3.64} = 474.72 \text{ lbs.}$$

When W rises 20 feet, then clearly P_1 will be displaced $3 \times 20 = 60$ feet.

\therefore Work done by $P_1 = 474.72 \times 60 = 28,483.2$ ft.-lbs.

And, Work done on $W = 1,000 \times 20 = 20,000$ ft.-lbs.

\therefore Work done against friction $= 28,483.2 - 20,000 = 8,483.2$ ft.-lbs.

We might also find the efficiency of this machine.

$$\text{Efficiency} = \frac{W y}{P_1 x} = \frac{m^2 + 3m + 3}{(1+m)^3} \times \frac{1}{3}$$

Where x = displacement of P_1 , and y = corresponding displacement of W , and $y : x = 1 : 3$, there being three parts of rope supporting W .

$$\therefore \text{Efficiency} = \frac{3.64}{3 \times 1.728} = .7021 = 70.21 \text{ per cent.}$$

Hence, 29.79 per cent. of total work expended is lost in friction.

LECTURE IV.—QUESTIONS.

1. State the principle of work, and apply it to show that a balanced lever whose arms are 2 and 3 will remain in equilibrium when weights which are as 3 and 2 are suspended at its ends.

2. Apply the principles of moments and of work in determining the relation between P and W in the wheel and compound axle. A weight of 20 lbs. draws up W lbs. by means of a wheel and compound axle. The diameter of the wheel is 5 feet, and the diameters of the parts of the compound axle are 9 and 11 inches respectively; find W. *Ans.* 1,200 lbs.

3. A compound axle consists of 2 parts, the diameters being 10 and 12 inches respectively, and a rope is coiled round them in opposite directions so as to form a loop, upon which hangs a pulley loaded to 48 lbs. Considering the parts of the rope to be vertical, find the force which, acting at a leverage of 4 feet upon the axle, will just balance the weight. Sketch the arrangement. *Ans.* $\frac{1}{2}$ lb.

4. In a compound wheel and axle, where the weight hangs on a single movable pulley, the diameters of the two portions of the axles are 3 and 2 inches respectively, and the lever handle which rotates the axle is 12 inches in length. If a force of 10 lbs. be applied to the end of the lever handle, what weight can be raised? *Ans.* 480 lbs.

5. Define the terms force ratio and velocity ratio as applied to machines. What must be the difference in the diameters of a compound wheel and axle so that the velocity of P may be 100 times that of W, the length of the handle being $2\frac{1}{2}$ feet? *Ans.* 1·2 inches.

6. In a compound wheel and axle, let the diameter of the large axle be 6 inches, and that of the smaller axle 4 inches, and the length of the handle 20 inches; find the ratio of the velocity of the handle to that of the weight raised. *Ans.* 40 : 1.

7. Define the terms, force ratio, velocity ratio, theoretical and actual advantages and efficiency of a machine. A tackle consists of two blocks, each weighing 10 lbs. The lower or movable block has two sheaves, and the upper or fixed one has three sheaves. It is found that a force of 56 lbs. is required to raise a weight of 200 lbs. suspended from the hook of the lower block. Find (1) the theoretical advantage, (2) the actual advantage, (3) the efficiency of the machine, (4) the percentage efficiency. If W rises 6 feet, what length of rope must be hauled in? *Ans.* (1) 4·76 : 1; (2) 3·57 : 1; (3) 71; (4) 71; 30 feet.

8. Describe Weston's differential pulley. If the weight is to be raised at the rate of 5 feet per minute, and the diameters of the pulleys of the compound sheave are 7 and 8 inches respectively, at what rate must the chain be hauled? *Ans.* 80 feet per minute.

9. State and explain the principle of the conservation of energy and show that the principle of work is only a particular case of this general principle.

10. State the principle of work and apply it to determine the relation between P and W in Weston's differential pulley block. In such a block the radii of the pulleys are 5 inches and $4\frac{1}{2}$ inches respectively. Taking the efficiency of the machine at 50 per cent.; what force must be applied to the hauling chain in order to raise a weight of 1 ton? What is the actual advantage in this machine? *Ans.* 224 lbs.; 10 : 1.

11. Explain the methods which you would adopt to find the mechanical advantage and efficiency of any machine, such as the ordinary block and tackle, or a Weston's differential block. Having found the theoretical pull (P) and the actual pull (Q) required to raise a given weight (W), what would be the efficiency of the machine? Give reasons for your answer. *Ans.* Efficiency = P/Q .

12. Given a Weston's pulley block, show how we find the velocity ratio. How would you experimentally find the real or mechanical advantage? Does the mechanical advantage depend on the load which is being lifted, and if so, in what way?

13. Sketch in section, and describe the construction of a differential pulley block working with an endless chain. Why with such a system of blocks does the weight remain suspended after the pull has been taken off the chain? Indicate clearly on your sketch the position of the chain on the pulleys and the snatch block, showing which side of the fall of the chain should be pulled in order to raise and lower the load respectively. If in a Weston's pulley block only 40 per cent. of the energy expended is utilised in lifting the load, what would require to be the diameter of the smaller part of the compound pulley when the largest diameter is 8 inches, in order that a pull of 50 lbs. on the chain may raise a load of 550 lbs.?
Ans. 7·4 inches.

14. Sketch a six-fold purchase tackle, and, denoting the velocity of the running block by unity, find the velocity of each ply of the rope.

(C. & G., 1902, O., Sec. A.)

N.B.—See *Appendices B and C for other questions and answers.*

LECTURE IV.—A.M. INST.C.E. EXAM. QUESTIONS.

1. Describe any machine for lifting weights. How do we find its velocity ratio? Describe very carefully how you would determine its real mechanical advantage under various loads and the sort of results you would expect to obtain. (I.C.E., Oct., 1898.)

2. In a certain machine the effort E , and R the resistance steadily overcome, were observed to have the following values:—

R . lbs.	14	28	42	56	70	84	98	112	126
E . lbs.	1.24	2.14	2.98	4.00	4.98	5.80	6.84	7.98	8.75

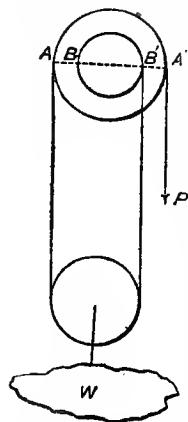
Find, by plotting on squared paper, the law connecting E and R . What is the mechanical advantage when R is 90 lbs. If for this value of R the efficiency is 70 per cent., what is the velocity ratio for the machine?

(I.C.E., Feb., 1899.)

3. Discuss the efficiency of a differential pulley-block, and explain exactly why it does not overhaul. (I.C.E., Oct., 1900.)

4. Show how to graduate the common steel-yard. If the sliding weight be 10 lbs. and the graduations for 1 stone and 2 stones be $3\frac{1}{2}$ inches apart, find how far from the fulcrum is the point at which the bodies weighed are attached. (I.C.E., Oct., 1900.) (See my "*Elementary Applied Mechanics*.")

5. In a wheel and axle $A A'$ and $B B'$ are the horizontal diameters; the force is applied at A' and the weight supported by two vertical ropes from A and B' . Neglect friction and determine the mechanical advantage. If there is friction at the bearings to such an extent that the weight will just not run down when no force is applied, show that the mechanical advantage when the force will just raise the weight W is $\frac{a+b}{2(a-b)}$, where a and b are the radii of the wheel and axle respectively. (I.C.E., Feb., 1901.)



6. The diameter of the opening of a safety valve is 4 inches; the distance from the fulcrum to the centre of the valve is 5 inches; the lever is 21 inches long; weighs $3\frac{1}{2}$ lbs.; and its centre of gravity is 8 inches from the fulcrum; the valve weighs 5 lbs. What weight must be hung 1 inch from the end of the lever so that steam may blow off at 50 lbs. absolute per square inch? (I.C.E., Feb., 1901.) (See my "*Elementary Applied Mechanics*.")

7. Explain with a sketch the construction and working of the differential chain-pulley block.

If the diameter of the larger groove is d_1 , and of the smaller groove d_2 , show that the velocity ratio in raising the load is $\frac{2d_1}{d_1 - d_2}$, and for lowering the load is $\frac{2d_2}{d_1 - d_2}$. Does the size of the chain affect these expressions, and if so how? Show that the mechanical

efficiency of such a machine is, $\frac{W}{P(VR)}$, where W = load lifted, P = force or effort, and VR = velocity ratio. Find the maximum mechanical efficiency (assuming the load to be infinitely great) of such a machine which has a velocity ratio of 15.8, and where the relation of P to W is $P = 2.4 + 0.18 W$. Explain why in such a machine the load does not run down or "overhaul" when the force or effort is removed. (I.C.E., Oct., 1901.)

8. In experimenting with a Weston differential pulley block, it is found that a force of 10 lbs. is required to raise a load of 50 lbs. and that a force of 18 lbs. will raise 100 lbs. The curve obtained by plotting these and other results with the forces as abscissæ and the loads as ordinates is found to be a straight line. Determine the constants in the equation of this line, and state what may be inferred as to the amount of friction in the contrivance. If the circumference of the larger wheel is 18 inches and the velocity ratio of the pulley is 16, what is the circumference of the smaller wheel? (I.C.E., Oct., 1902.)

9. The safety valve of a boiler is required to blow off steam at 100 lbs. per square inch by gauge. The dead weight is 100 lbs., weight of lever 10 lbs., and of valve 5 lbs.; diameter of valve $3\frac{1}{2}$ inches; distance from centre of valve to fulcrum 4 inches; from centre of gravity of lever to fulcrum 15 inches. How far from the fulcrum should the weight be placed on the lever? (I.C.E., Feb., 1903.) (See my "*Elementary Applied Mechanics*.")

N.B.—See Appendices B and C for other questions and answers.

LECTURE V.

CONTENTS. — Definition of Friction—Limiting Friction—Definition of Coefficient of Friction—Static and Kinetic Friction—Ordinary Laws of Friction for Plane Surfaces—Morin's Experiments—General Results of Recent Experiments on the Friction of Plane Surfaces—Simple Methods for Finding the Coefficients of Friction and Angles of Repose—Definition of Angle of Repose—Limiting Angle of Resistance and its Definition—Examples I. and II.—The best Angle of Propulsion or Traction—Example III.—Questions.

DEFINITION.—Friction is the term used to denote the resistance to motion which is experienced when one body is made to slide over the surface of another.

The true cause of friction is the roughness of the surfaces in contact. The surfaces of all bodies are more or less rough, and, when examined by means of a microscope, they are found to be covered with minute projections, which are smaller the smoother the surface. When one surface rests upon another, the projec-



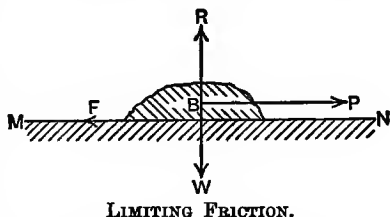
MAGNIFIED SECTION THROUGH TWO
ROUGH SURFACES IN CONTACT.

tions of the one appear to fit into corresponding hollows in the other. Hence, to move the one surface relatively to another a certain force must be exerted either in separating (*i.e.*, lifting) the surfaces sufficiently to clear these projec-

tions; or, in breaking off some and clearing others. By interposing a lubricant, such as oil or grease, between the surfaces, the friction may be greatly diminished. In such cases, the surfaces do not appear to be in actual contact, but are separated by a thin film of the lubricant, over which they slide. The amount by which the friction is thus diminished depends on the nature and quantity of the lubricant between the rubbing surfaces.

Limiting Friction.—Friction is thus a *tangential* resistance offered to the motion of one body over the surface of another. Thus, if the body, B, is made to slide along the surface, M N, by the force, P, the frictional resistance, F, always acts along the common tangent to the two surfaces in contact. Whilst B is

just beginning to move, the resistance, F , increases from nothing to a certain limit, so that any further increase of P causes the body to slide. The greatest amount of friction thus called into play is usually spoken of as the **Limiting Friction**. It depends for its magnitude on the reaction, R , between the surfaces in contact (due to the weight of the body, W) and the nature of those surfaces. The limiting friction, F , is *measured* by the least force, P , which just causes sliding to take place in a horizontal plane.



DEFINITION.—The Coefficient of Friction (μ) is the ratio of the Limiting Friction, F , to the Normal Reaction, R , between the surfaces in contact.

$$\text{i.e.,} \quad \mu = \frac{F}{R}; \text{ or, } F = \mu R$$

Static and Kinetic Friction.—It has been proved experimentally that the “limiting friction” between surfaces at rest relatively to each other, is slightly different in magnitude from that between the same surfaces when in motion. The former has been called **Static Friction** or **Friction of Rest**, whilst the latter is called **Kinetic Friction** or **Friction of Motion**.

Ordinary Laws of Friction for Plane Surfaces.—In 1785, Coulomb, a French officer, published the results of a series of experiments carried out by him on the friction of plane surfaces. These results he embodied in the following statements, usually called the ordinary laws of friction :—

LAW I.—The friction between two bodies is directly proportional to the normal pressure between them.

LAW II.—The friction is independent of the areas of the surfaces in contact.

LAW III.—Kinetic friction is less than static friction, and is independent of velocity.*

It will at once be seen, that these three laws may be comprised in the single statement that, The coefficient of friction depends only on the nature of the surfaces in contact.

* For experimental methods of verifying these laws see the Author's *Elementary Manual on Applied Mechanics*.

Morin's Experiments.—Coulomb's experiments were not considered sufficiently extensive to thoroughly establish the truth of the above so-called laws. The whole subject has, however, been reinvestigated by several persons, notably by General Morin during the years 1831-34. The results of Morin's experiments were, for a long time, regarded as conclusively establishing the above laws. This was no doubt true within the limits of the pressures and the velocities he employed; but, in some recent experiments, which have been carried out with much greater care and wider variations, both in pressures and velocities, the laws of Coulomb were found to be erroneous. The coefficients of friction, instead of being independent of pressure and velocity, are shown to vary considerably with the pressure, velocity, and temperature.

General Results of Recent Experiments on Friction of Plane Surfaces.—(1) With *dry* surfaces the coefficient of friction *increases* with the intensity of the pressure. The highest pressure employed by Morin was little more than 100 lbs. per square inch, and it is just about this pressure that deviation from Coulomb's law appears to begin. This increase in the friction with high pressures is probably due to abrasion of the surfaces; but, when the same surfaces were well lubricated the reverse took place.

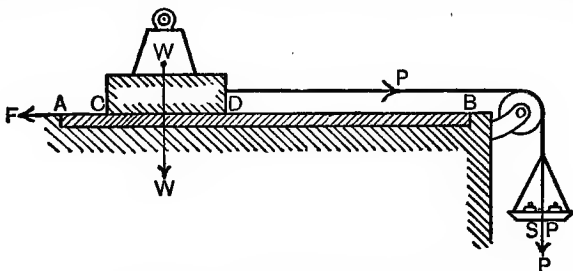
(2) The lowest pressure employed by Morin was about $\frac{3}{4}$ lb. per square inch, but in recent experiments with pressures lower than this, the coefficient of friction was found to *increase* as the pressure *decreased*.

(3) With *high* velocities the coefficient *diminishes* as the velocity *increases*. These results are only true with velocities greater than those employed by Morin.* With all velocities under 10 feet per second it has recently been found that the coefficient of friction is quite independent of the speed.

Simple Methods for Finding the Coefficients of Friction and Angles of Repose.—(1) Take two pieces of the materials to be tested. Let one of these, A B, be shaped like a lath and laid on a table, while the other, C D, is made to slide on its upper surface as shown by the figure. The bodies are pressed together by a weight, W, which may be varied at pleasure. The reaction, R, between the two surfaces will be $W + \text{weight of block, C D}$. Now load the scale pan, S P, with small shot until the block, C D, moves freely along A B. The motion may require to be aided by a little tapping on the table, since static friction is greater than kinetic friction. The experiment should be re-

* 10 feet per second was the highest velocity in Morin's experiments.

peated two or three times, taking care that P , the force causing motion, is not more than is necessary to just keep CD moving

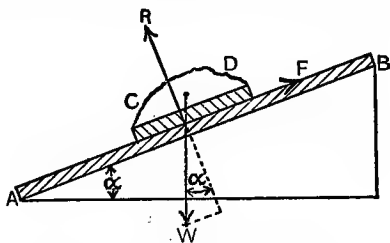


FINDING THE COEFFICIENT OF FRICTION.

at a *uniform* rate. Next remove SP and weigh it carefully. Let this weight be P units.*

Then, The coefficient of friction $= \mu = \frac{F}{R} = \frac{P}{R}$.

(2) Incline the plane AB gradually until CD just begins to slide downwards, the table being gently tapped to overcome the



FINDING ANGLE OF REPOSE.

static friction. Let α be the angle which the plane AB now makes with the horizontal, then :—

The coefficient of friction $= \mu = \tan \alpha$.

For, resolve W along AB and at right angles to AB . Then:—

The component along $AB = F = W \sin \alpha$.

And the component at }
right angles to AB , } $= R = W \cos \alpha$.

$\therefore \mu = \frac{F}{R} = \frac{W \sin \alpha}{W \cos \alpha} = \tan \alpha$.

* If P be in pounds, ounces, grammes, or grains, R must also be in pounds, ounces, grammes, or grains.

The angle α is called the "*angle of repose*," or "*angle of friction*" for the materials A B and C D. Hence:—

DEFINITION.—The angle of repose is the greatest angle at which a plane may be inclined to the horizon before a body placed on it will begin to slide.

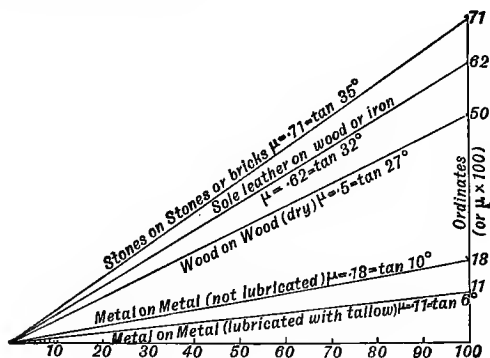
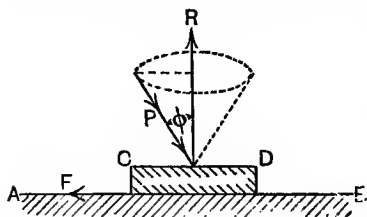


DIAGRAM OF ANGLES OF REPOSE.

The above diagram exhibits the "angles of repose" for several of the more common materials, together with the values of their coefficients of friction.

Limiting Angle of Resistance.—If we attempt to push the block, C D, along A B by means of a sharp pointed rod inclined to the vertical as shown; then, it will be found that, however great the pressure exerted, no relative motion will take place unless the rod be inclined to the vertical at an angle at least equal to the angle of repose.

Thus, let ϕ be the angle which the direction of the force, P, makes with the normal reaction, R, when sliding is just about to take place. Resolve P parallel and perpendicular to A B. Then, clearly, the limiting friction, F, between A B and C D is equal to the component of P parallel to A B—i.e., $F = P \sin \phi$. Also, the perpendicular pressure between the surfaces (neglecting the weight of C D) is $R = P \cos \phi$.



LIMITING ANGLE OF RESISTANCE.

$$\therefore \mu = \frac{F}{R} = \frac{P \sin \phi}{P \cos \phi} = \tan \phi.$$

But we have just seen that $\mu = \tan \alpha$.

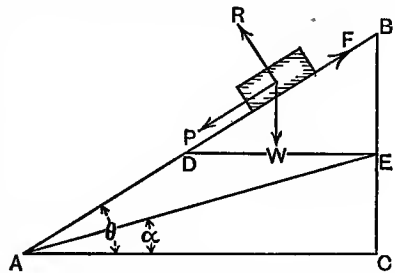
$$\therefore \mu = \tan \phi = \tan \alpha. \quad \text{Or, } \phi = \alpha.$$

Hence, when a body is made to slide along the surface of another, the direction of the total reaction makes an angle with their common normal (at least) equal to the angle of repose. This angle is called the *limiting angle of resistance* and may be thus defined.

DEFINITION.—The limiting angle of resistance is the greatest angle which the total reaction between two surfaces can make with the normal before sliding takes place.

(3) Another method for finding the average coefficient of friction is the following:—

Take a plane, A B, made of one substance and inclined at any angle, θ (greater than the angle of repose). Allow a block of the other substance to slide along a given length, B A, of the inclined plane, and note its velocity when it reaches the point, A. Next calculate the *vertical* height, B E, corresponding to the length, B D, of the plane through which the body would require to acquire the *same* velocity as before, *if there was no friction*.



FINDING THE COEFFICIENT OF FRICTION.

To get this height, B E, let v be the actual velocity of the body at A. Then, neglecting friction, this velocity would be acquired when the body reached the point D, such that:—

$$v^2 = 2g \times BE, \therefore BE = \frac{v^2}{2g}$$

Set off this distance along B C. Join A E.

Then :—Average coeff. of frict. from B to A = $\frac{EC}{AC} = \tan \alpha$.

Let B C represent the *total* force, P, impelling the body down, B A, against friction and generating the velocity, v , then

* See Contents for Lecture on Motion.

BE will represent the force which goes to generate the velocity alone, since in the second case the final velocity is the same as in the first and, by hypothesis, no frictional resistances are overcome. Hence, the force which overcomes the friction alone, will be represented by the difference between BC and BE—i.e., by EC.

Now, we know that:—

$$P : R = BC : AC.$$

Hence,

$$F : P : R = EC : BC : AC.$$

$$\therefore \mu = \frac{F}{R} = \frac{EC}{AC}.$$

The chief difficulty in making an experiment of this kind would be in finding the velocity, v , at A. It is much easier to find the time taken to move from B to A. Suppose this time to be found. Let it be, t , seconds. Then assuming the body to be uniformly accelerated, we get $s = \frac{1}{2} v t$, or $AB = \frac{1}{2} v t$.*

$$\therefore v = \frac{2 AB}{t}.$$

EXAMPLE I.—Let the plane, AB, be 10 feet long and inclined to the horizon at an angle of 30° . Suppose the time taken to slide from B to A to be $1\frac{1}{2}$ seconds. Then the velocity at the foot of the plane is $v = \frac{2 \times 10}{1.5} = \frac{40}{3}$ feet per second.

$$\text{The height BE} = \frac{v^2}{2g} = \frac{\frac{40}{3} \times \frac{40}{3}}{2 \times 32.2} = 2.76 \text{ feet.}$$

Then, since angle BAC = 30° , $\therefore BC = \frac{1}{2} AB = 5$ feet.

$$\therefore CE = 5 - 2.76 = 2.24 \text{ feet.}$$

$$\text{Also, } AC = \sqrt{AB^2 - BC^2} = \sqrt{75} = 8.66.$$

$$\therefore \mu = \tan \angle EAC = \frac{EC}{AC} = \frac{2.24}{8.66} = .2586$$

and this corresponds to an angle of friction of 15° nearly.

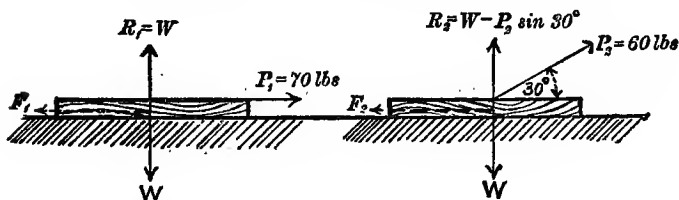
This method of obtaining the coefficient of friction, although both interesting and instructive, is, however, so complicated and attended with so many difficulties that reliable results could not be obtained in a class-room or ordinary laboratory.

EXAMPLE II.—A plank of oak lies on a floor with a rope

* See Lecture on Motion, &c., in this treatise.

attached to it. When the rope is pulled horizontally with a force of 70 lbs. it just moves, but when pulled at an angle of 30° to the floor a force of 60 lbs. moves it. What is the weight of the plank and the coefficient of friction between it and the floor?

ANSWER.—Let W denote the weight of the plank in lbs.,
 „ μ „ coefficient of friction between the
 floor and the plank.
 „ F_1, F_2 „ frictions in the two cases.



EXAMPLE ON COEFFICIENT OF FRICTION.

In the first case, when P is parallel to the floor, we get $R_1 = W$, and $P_1 = F_1 = \mu R_1$,

$$\therefore \mu W = 70 \quad \dots \quad (1)$$

In the second case, when P is inclined at an angle of 30° to the floor, the reaction between the floor and the plank will be less than W by the vertical component of P_2 .

$$\therefore R_2 = W - P_2 \sin 30^\circ = W - 60 \times \frac{1}{2} = (W - 30) \text{ lbs.}$$

$$\therefore F_2 = \mu R_2 = \mu (W - 30) \text{ lbs.}$$

$$\text{But } F_2 = P_2 \cos 30^\circ = 30 \sqrt{3} \text{ lbs.}$$

$$\therefore \mu (W - 30) = 30 \sqrt{3} \quad \dots \quad (2)$$

We have now obtained two equations, (1) and (2), containing the two unknown quantities, W and μ . By solving these equations these quantities can be found.

Divide (2) by (1), then

$$\frac{W - 30}{W} = \frac{30 \sqrt{3}}{70} = \frac{3 \sqrt{3}}{7}.$$

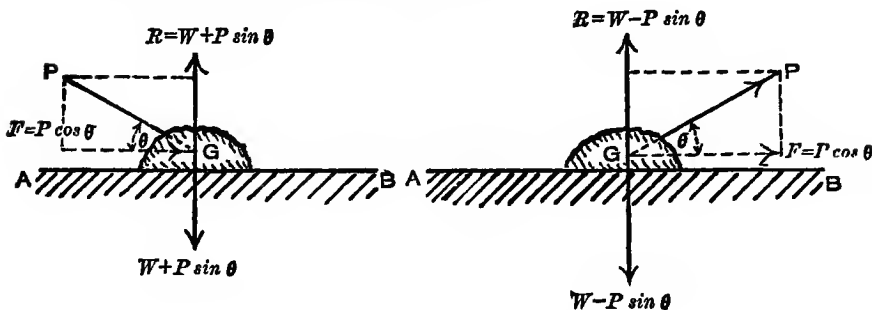
$$\therefore 7W - 210 = 3 \sqrt{3} W.$$

$$\therefore W = \frac{210}{1.8} = 116.6 \text{ lbs. nearly.}$$

$$\text{From equation (1), } \mu = \frac{70}{W} = .6 \text{ nearly.}$$

From this example it appears that the pull, P (which is sometimes called the *traction*), *decreases* as its direction becomes more inclined to the line of motion. Thus, when P is parallel to the floor, the pull required is 70 lbs., but when inclined at an angle of 30° it is only 60 lbs. It is also clear that P does not continually decrease as the *angle of traction* increases. For, when P makes an angle of about 90° with the line of motion, the tendency of P is to lift W and not to move the body along the floor at all. Hence, there must be some definite angle for which the pull, P , will have its minimum value.

The Best Angle of Propulsion or Traction. — We shall now show that the best angle of propulsion or traction for given materials is equal to their angle of repose.



ANGLE OF PROPULSION OR TRACTION.

Let P make an angle, θ , with the direction of motion. Let α be the angle of repose for the two materials. Then $\mu = \tan \alpha$.

Normal pressure between the bodies = $R = W \pm P \sin \theta$.*

Resistance to motion = $F = \mu R = \mu (W \pm P \sin \theta)$

Force causing motion = $P \cos \theta$.

$$\therefore P \cos \theta = \mu (W \pm P \sin \theta)$$

$$\therefore P (\cos \theta \mp \mu \sin \theta) = \mu W.$$

* The sign is + when P is pushing the body, as shown by the left-hand figure; and - when P is pulling the body, as shown by the right-hand figure. Hence, throughout the following investigation the *upper* sign will refer to the left-hand figure and the *lower* sign to the right-hand figure.

$$\text{Or, } P = \frac{\mu W}{\cos \theta \mp \mu \sin \theta}. \quad \text{But, } \mu = \frac{\sin \alpha}{\cos \alpha}.$$

$$\text{Hence, } P = \frac{W \sin \alpha}{\cos \theta \cos \alpha \mp \sin \theta \sin \alpha}, \therefore P = \frac{W \sin \alpha}{\cos (\theta \pm \alpha)} \quad (1)$$

Hence, P will have its *minimum* value, for a given load, W , when the fraction $\frac{\sin \alpha}{\cos (\theta \pm \alpha)}$ is a *minimum*. Now, the denominator of this fraction is the only quantity which can be made to vary, since α , and therefore also, $\sin \alpha$ is a constant quantity for the same materials. Consequently, the fraction will be a *minimum* when its denominator is a *maximum*—i.e., when $\cos (\theta \pm \alpha)$ is a *maximum*. But the maximum value of a cosine is *unity*, and this occurs when the angle is 0.

\therefore When $\cos (\theta \pm \alpha) = 1$, $\theta \pm \alpha = 0$, or $\theta = \mp \alpha$.*

Hence, the least push or pull required to move a load, W , along a horizontal plane is, by equation (1),

$$P = W \sin \alpha$$

and the direction of the push or pull makes an angle α , equal to the angle of repose, with the horizontal plane.

EXAMPLE III.—A body weighing 200 lbs. is drawn along a horizontal plane, by a rope making an angle of 30° to the plane. Find the force necessary to move the body, supposing the coefficient of friction to be .5. Find, also, the least force which would just suffice.

ANSWER.—(1) Referring to the previous right-hand figure, we get:—

$$P \cos 30^\circ = F = \mu R = \mu (W - P \sin 30^\circ)$$

$$\therefore P \times \frac{\sqrt{3}}{2} = .5 (200 - P \times \frac{1}{2})$$

$$\therefore .866 P = 100 - .25 P$$

$$\therefore P = \frac{100}{1.116} = 89.6 \text{ lbs.}$$

* The upper or (−) sign, here refers to the case wherein the body is being *pushed*, while the lower or (+) sign refers to the case wherein the body is being *pulled*. In the first case, we see that P will be a *minimum* when $\theta = -\alpha$; i.e., when the force, P , is directed from *below upwards* and inclined to the horizon at an angle, α , equal to the “angle of repose.” Similarly, the *pull*, P (right-hand figure), will be a *minimum* when $\theta = +\alpha$, —i.e., when P is directed upwards and inclined to the horizon at an angle, α , equal to the “angle of repose.”

(2) The least force necessary to move the body is, according to the above results,

$$P = W \sin \alpha.$$

Now, $\tan \alpha = \mu,$

$$\therefore \frac{\sin^2 \alpha}{\cos^2 \alpha} = \mu^2.$$

Or, $\sin^2 \alpha = \mu^2 (1 - \sin^2 \alpha),$

$$\therefore \sin \alpha = \frac{\mu}{\sqrt{1 + \mu^2}} = \frac{.5}{\sqrt{1.25}} = .447$$

$$\therefore P = 200 \times .447 = 89.4 \text{ lbs.}$$

In this case the direction of P makes an angle of $26\frac{1}{2}^\circ$ nearly with the horizontal plane.

The student should now prove that the same holds true when a body is pulled up an inclined plane by a force, P , which makes an angle, θ , with the incline—viz., that the force will be least when $\theta = \alpha$, the angle of repose.

LECTURE V.—QUESTIONS.

1. What is friction? State the ordinary laws of friction, and explain by aid of sketches and concise descriptions how they may be proved experimentally. What is meant by the "coefficient of friction;" "angle of repose;" "angle of friction;" and "limiting angle of resistance?"

2. Define the coefficient of friction and the angle of friction. A weight of 500 lbs. is placed on a table, and is just made to slide by a horizontal pull of 155 lbs. Find the coefficient of friction, and the number of degrees in the angle of friction, by drawing it to scale. *Ans.* 0·31.

3. The saddle of a lathe weighs 5 cwts., and it is moved along the bed of the lathe by a rack and pinion arrangement. What force, applied at the end of a handle 10 inches in length, will be just capable of moving the saddle, supposing the pinion to have 12 teeth of $1\frac{1}{4}$ -inch pitch, and the coefficient of friction between the saddle and lathe-bed to be 0·1, other friction being neglected? Sketch the arrangement. *Ans.* 13·36 lbs.

4. A body weighing 50 lbs. is pulled along a rough horizontal plane by a force whose line of action makes an angle of 45° with the plane. If the coefficient of friction between the body and the plane be 0·2, find the magnitude of the pull and the pressure between the body and the plane. *Ans.* 11·78 lbs.; 41·6 lbs.

5. A body is resting on a rough horizontal plane, and is acted on by a force whose line of action is inclined 45° to the plane. The force is gradually increased until the body is just about to move; find the ratio of the force exerted, to the weight of the body, the coefficient of friction being 0·25. *Ans.* $\sqrt{2} : 5$, or $1 : 3\cdot5$.

6. A body lying on a rough table can just be moved by a horizontal pull of 20 lbs.; but, when pulled at an angle of 30° to the horizon, the force required is found to be only 18 lbs. Will you explain the reason for this difference, and find the weight of the body and the coefficient of friction between it and the table? *Ans.* 41 lbs.; ·49.

7. A body weighing 100 lbs. is drawn along a horizontal plane by a rope, making an angle of 20° with the plane. Find the force required, supposing the coefficient of friction to be 0·15. Find, also, the least force which would just pull the body along the plane, and the angle which its direction would make with the plane. *Ans.* 15 lbs.; 14·8 lbs.; $8\frac{1}{2}^\circ$.

8. A body of known weight is placed on a rough horizontal plane and pulled in a certain direction. Find (1) the force of the pull which will just make the body slide, and (2) what must be the direction of the pull that it may be the least that will make the body slide? Suppose that P is the least pull as above determined, and that the body when pushed by a force, P_1 (acting along the same line as that in which P acted), is on the point of sliding, show that $P_1 (1 - \mu^2) = P (1 + \mu^2)$, μ being the coefficient of friction.

9. A force of 15 lbs. per ton of load, is required to maintain the motion of a train on a level line. Determine the coefficient of friction between the driving-wheels and rails when an engine of 30 tons weight can just keep in motion a train of 350 tons (including weight of engine). Give a diagram illustrating the method employed by you in arriving at the answer. *Ans.* 0·078.

10. Show how you would arrange and use an apparatus for determining the coefficient of friction between iron and brass. We particularly wish to know how the friction depends upon the speed of rubbing. (B. of E. H., Part I., 1902.)

N.B.—See Appendices B and C for other questions and answers.

LECTURE V.—A.M.INST.C.E. EXAM. QUESTIONS.

1. Define the terms coefficient of friction and angle of repose, and show how these quantities are related to one another. A ladder rests against a wall at such an angle that it is nearly slipping down. How is its stability affected when a man goes up it? (I.C.E., *Oct.*, 1897.)

2. The coefficient of friction between a block and a plane surface is $\frac{1}{4}$. A cord leads away from the block parallel to the surface and over a pulley at the end. The cord is loaded with a weight of 5 lbs. and the weight of the block is 10 lbs. Neglecting the friction of the pulley, find at what inclination the surface must stand if the block is (1) on the point of slipping down, (2) on the point of slipping up. (I.C.E., *Feb.*, 1898.)

3. A body of 50 lbs. is on an inclined plane of inclination 35° . The coefficient of friction is 0.16. A force x acts upon the plane, making an angle of 10° with the plane (45° with the horizontal). Working either graphically or arithmetically, find x . First, when it just allows the body to slip down the plane; second, when it pulls the body up the plane. (I.C.E., *Oct.*, 1898.)

4. A square block weighing 40 lbs. rests on a flat horizontal surface. A horizontal force is applied to the block, the amount of the force increasing from 0 lb. to 8 lbs., the block being then just on the point of moving. Show in a diagram the amount and direction of the reaction of the surface on the block (1) when the applied force is 4 lbs., and (2) when it is 8 lbs. Now suppose the direction of the applied force is altered so that it makes an angle of 20° with the horizontal; again, show the reaction of the surface on the block when the applied force is 6 lbs. weight. What must this force be increased to so as just to cause a steady motion? Find the least force necessary to move the block, giving both its magnitude and direction. (I.C.E., *Feb.*, 1899.)

5. A uniform beam stands on a rough horizontal floor and leans against a vertical wall of equal roughness, find the relation which must subsist between the angle of friction and the inclination of the beam to the horizontal, if the latter is on the point of moving. (I.C.E., *Oct.*, 1899.)

6. State the laws of statical and sliding friction. If the coefficient of friction be $\frac{1}{3}$, find the least depth, from back to front, of a drawer, of width 2 feet, which can be drawn out by a direct pull on a handle 6 inches to the right or left of the middle of the front. (I.C.E., *Feb.*, 1901.)

7. A body rests upon a rough inclined plane, and the plane is inclined to the horizontal, until the body is just about to slide down the plane. State the forces acting on the body, and express the relations between them. Show that the tangent of the angle at which sliding takes place is equal to the coefficient of friction between the sliding surfaces. (*Oct.*, 1901.)

8. State the laws of dry or solid friction, and describe how they may be verified by experiment. A loaded wood slider weighs 10 lbs., and just rests without sliding upon a wood inclined plane, the base of which is 28.9" and the height 6.4". Find the coefficient of friction. (I.C.E., *Feb.*, 1902.)

9. A man ascends a ladder resting on a rough horizontal floor against a smooth vertical wall. Determine, graphically or otherwise, the direction of the section between the foot of the ladder and the floor. (I.C.E., *Feb.*, 1903.)

10. A plane inclined at 20° to the horizontal carries a load of 1,000 lbs., and the angle of friction between the load and plane is 10° . Obtain the least force in magnitude and direction which is necessary to pull the load up the plane. (I.C.E., *Oct.*, 1903.)

11. A ladder, whose centre of gravity is at the middle of its length, rests on the ground and against a vertical wall; the coefficients of friction of the ladder against both being $\frac{1}{4}$. Find the ladder's inclination to the ground when just on the point of slipping. (I.C.E., *Oct.*, 1904.)

N.B.—See Appendices B and C for other questions and answers.

LECTURE VI.

CONTENTS.—Friction of Cylindrical Surfaces—General Morin's Experiments—Hirn's Experiments—Prof. Thurston's Experiments—Prof. Fleeming Jenkin's Experiments—Beauchamp Tower's Experiments on Journals—Practical Examples of Lubricating Journals—Heavy Shaft Bearings—Eccentric Bearings—Self-Oiling Bearings—Adjustable Bearings—Automatic Engine Lubrication—The Seigrist System of Automatic Lubrication for Large Engines—Experiments on Collar Friction—Friction of a Pivot Bearing—Results of the Experiments—Friction of Railway Brakes—Friction between Water and Bodies Moving through it—Frictional Resistance of a Ship Propelled through Sea Water—Examples—Questions.

Friction of Cylindrical Surfaces.—In the preceding lecture we only dealt with the friction of plane surfaces. It is equally necessary, however, that the engineer should study the friction of cylindrical surfaces. With this object in view we shall give a brief summary of the results of the experiments carried out by the principal authorities on this subject prior to the year 1883, and then state the conclusions arrived at by the "Research Committee on Friction," appointed by the "Institution of Mechanical Engineers," which now constitute the standard and most reliable experiments on this subject.

General Morin's Experiments.—Morin also made experiments on the friction of axles, and he arrived at the same general conclusions as were explained in Lecture V. for plane surfaces; the only difference being, in the values of the coefficients of friction. The diameters of his journals reached a maximum of 4 inches, but the speeds never exceeded a sliding velocity of more than 30 feet per minute, and the pressures 160 lbs. per square inch of the nominal bearing surface. By nominal bearing surface is meant, the projected area of the journal on a diametrical plane—i.e., on a plane containing the axis of the journal. Thus, let d denote the diameter and l the length of the journal in inches, then :—

Nominal bearing surface = dl square inches.

If R be the total reaction or load in pounds, and p the intensity of pressure, or the pressure in pounds per square inch on the journal, then :—

$$R = p dl; \text{ or } p = \frac{R}{dl}.$$

* If the arc of the circle embraced by the brass bush, upon which the shaft actually bears, be indicated by the chord, d' , subtended by it, then :—

$$R = p d' l, \text{ or } p = \frac{R}{d' l}.$$

The values of μ in the equation $F = \mu R$, as given by Morin, are:—for dry journals .18 to .25, for those greased and wet with water, .14 to .19; intermittently lubricated, .07 to .12; and for continuous lubrication, .03 to .05. *The friction was found to be independent of the velocity and proportional to the load.* The coefficient of friction thus depending *only* on the nature of the bearing surfaces. In more recent experiments with cases approaching those which occur in actual practice, it has been shown that the values of μ are much smaller than those given by Morin, and, further, that the friction is entirely dependent on the more or less thorough lubrication of the bearing.

Hirn's Experiments.*—In 1885, M. Hirn published the results of a long series of experiments, chiefly on lubricated journals. These results show, that instead of the coefficient of friction being a constant quantity for the same materials, it is more nearly proportional to the square root of the velocity of rubbing, v , and inversely proportional to the square root of the intensity of pressure, p .

$$\text{Or,} \quad \mu = c \sqrt{\frac{v}{p}}$$

Where c is a constant quantity found by experiment. Hence, we see that the friction in those experiments varied directly as the square root of the load, area, and velocity.

$$\text{For, } F = \mu R = c \sqrt{\frac{v}{p}} \times R = c \sqrt{\frac{v}{R}} \times R = c \sqrt{R} \frac{dv}{dl}$$

For ordinary conditions of working the friction thus appears to have varied as the square root of the velocity. The friction diminished as the temperature increased,† and the best results were obtained after the lubricant had been working for some time between the surfaces.

* In 1880, C. J. H. Woodbury, of the Institute of Technology, Boston, Mass., U.S.A., read a paper before the American Society of Mechanical Engineers (see vol. i., p. 74, *et seq.*) on "Measurement of the Friction of Lubricating Oils." He states that his experiments proved that the coefficient of friction varies in an inverse ratio with the pressure for high speed lightly-loaded spindles. Further, that the coefficient of friction at 150° F. was about 75 per cent. less than at 75° F.; and, therefore, mill owners should keep their machinery warm in winter.

† See a paper by M. G. Adolphus Hirn, read before the Société Industrielle de Mulhouse, June 28, 1884, where water is used to control the temperature of the bearing surfaces of oil testing machines; also, The 1886 Cantor Lectures on "Friction," by Prof. Hele Shaw.

Prof. Thurston's Experiments.*—Professor R. H. Thurston, of U.S.A., has carried out a number of experiments to determine the effect of changes, not only in velocity, but also in pressure and temperature, upon the frictional resistance of lubricated bearings. His conclusions are, that the coefficient at first decreases, but after a certain point increases with the velocity; the point of change varying with the pressure and temperature. Very few details, however, are given of the way in which these experiments were carried out, and, consequently, we cannot here enlarge further upon them.

Prof. Fleeming Jenkin's Experiments.†—A number of experiments was carried out by Prof. Jenkin in connection with the difference between Static Friction or the Friction of Rest and Dynamic Friction or the Friction of Motion. He experimented at extremely low velocities, and showed, that in certain cases, the coefficient of friction decreases gradually as the velocity increases, between speeds of $\cdot 012$ and $\cdot 6$ foot per minute, thus indicating the probability of a *continuous* rather than a sudden change in the value of the coefficient of friction between the conditions of rest and motion.

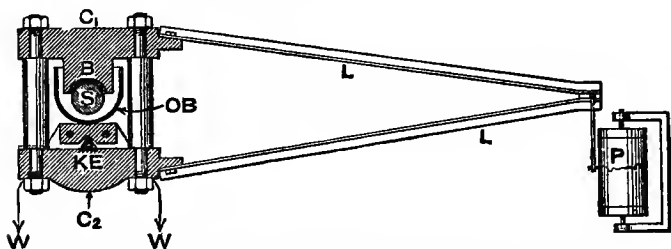
Beauchamp Tower's Experiments‡—(1) *Description of Machine.*—In experimenting on the friction of lubricated bearings, and on the value of different lubricants, one of the difficulties which is first met with is the want of a method of applying the lubricant, which can be relied upon as sufficiently uniform in its action. All the common methods of lubrication are so irregular in their action that the friction of a bearing often varies considerably. This variation, though small enough to be of no practical importance, and to pass unnoticed, in the working of an ordinary machine, would be large enough utterly to destroy the value of a set of experiments, say, on the relative values of various lubricants; for it would be impossible to know whether an observed variation was due to a difference in the quality of the oil, or in its rate of application. The first problem, there-

* "Friction and Lubrication," p. 185. "American Association for the Advancement of Science," Aug. 1878, p. 61. "The Theory of the Finance of Lubrication and on the Valuation of Lubricants by Consumers," "Friction and Lost Work in Machinery," N.Y., 1885, and on "The Real Value of Lubricants," Jany. 5th, 1885, see *Trans. Am. Soc. Mech. Engs.*, vol. xiii. Also see the 1891 vol. for "Special Experiments with Lubricants," by B. J. E. Denton of Hoboken, N.J., U.S.A. He deals with the lubrication of steam cylinders and of journals subjected to heavy pressures.

† *Proceedings of the Royal Society*, 1877, p. 93.

‡ By the kindness of the Council of the Institution of Mechanical Engineers, London, the author is permitted to make the following extracts from their Proceedings and Report on Friction Experiments, Nov. 1883.

fore, which presented itself, in the present experiments, was to devise a method of lubrication such as would be perfectly uniform in its action, and would form an easily reproducible standard with which to compare other methods. These conditions were best fulfilled by making the bearing run immersed in a bath of oil. By this method the bearing is always supplied with as much oil as it can possibly take; so that it represents the most perfect lubrication possible, and is a good standard with which to compare other methods. It is at all times perfectly uniform in its action. It is very easily defined and reproduced; and it also has the advantage that the temperature of the bearing can be easily regulated by gas jets under the bath. Experiment showed that the bath need not be full; the results obtained were the same when it was so nearly empty that the bottom of the journal only just touched the oil.



BEAUCHAMP TOWER'S APPARATUS FOR TESTING THE FRICTION OF JOURNALS.

The above figure represents the arrangements for conducting the experiments. The shaft, S, was of steel, 4 inches diameter and 6 inches long, with its axis horizontal and driven by a belt acting on a pulley keyed to its outer end.

A gun-metal brass, B, embracing somewhat less than half the circumference of the journal, rested on its upper side. The exact arc of contact of this brass was varied in the different experiments. Resting on this brass was a cast-iron cap, C₁, from which was hung by two bolts a cast-iron cross-bar, C₂, carrying a knife-edge, K E. The exact distance of this knife-edge below the centre of the journal was 5 inches. On this knife-edge was suspended the cradle which carried the weights, W, W, applied to the bearing. The cap, bolts, and cross-bar were put together in such a manner as to form a rigid frame, connecting the brass with the knife-edge. If there had been no friction between the brass and the journal, the weight would have caused the knife-edge to hang perpendicularly below the

axis of the journal. Friction, however, caused the journal to tend to carry the brass and the frame to which it was attached, round with it, until the line through the centre of journal and the knife-edge, made such an angle with the perpendicular, that the weight multiplied by the distance from the knife-edge to that perpendicular, offered an opposing moment just equal to the moment of friction.

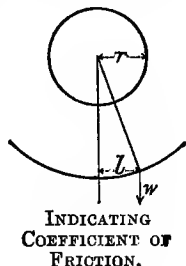
Suppose r = radius of the journal.

„ l = distance of the knife-edge from the perpendicular.

„ w = the weight.

And,

$(l \times w)$ = the moment of friction.

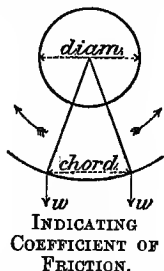


The friction at the surface } = $\frac{\text{the moment}}{r} = \frac{w \times l}{r}$.
of the journal,

Hence, the coefficient of } = $\frac{\text{Friction at surface of journal}}{w} = \frac{l}{r}$.
friction,

So that the coefficient of friction is indicated by l in terms of r , no matter what the weight is. As an example, suppose l was equal to r , the coefficient of friction would obviously be 1; or if l was $\frac{1}{10}$ of r , then the coefficient of friction would be $\frac{1}{10}$.

In order to avoid the difficulty of determining accurately when the knife-edge was perpendicularly under the centre of the journal (a knowledge which was necessary in order to obtain a measurement of l , and which was very difficult to obtain owing to the considerable friction between the brass and the journal when at rest), each experiment was tried with the journal revolving in both directions, and the sum of the values of l on both sides was measured; and then the coefficient of friction was indicated by the chord of the whole angle, included between the two lines of inclination caused by the friction, with the rotation in the two directions, the chord being expressed in terms of the diameter of the journal (see figure). Each result was thus a mean of two experiments, one with the axle running in one direction, and the other with it running in the other direction. In order to read the value of the co-



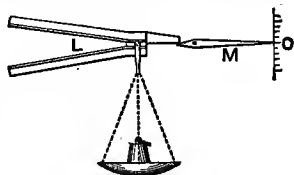
efficients thus obtained, a light horizontal lever, L, was attached to the frame connecting the brass to the knife-edge. It was $62\frac{1}{2}$ inches long, or twelve and a-half times the distance between the centre of the journal and the knife-edge; so that, at the end of the lever, the chord indicating the coefficient of friction was magnified twelve and a-half times. As a chord of 4 inches at the knife-edge represents a coefficient of 1, a chord of 50 inches at the end of the lever also represents a coefficient of 1, while 5 inches represents a coefficient of $\frac{1}{10}$, $\frac{1}{2}$ -inch of $\frac{1}{100}$, and $\frac{1}{20}$ -inch of $\frac{1}{1000}$. The position of the end of the lever during each experiment was recorded by a tracing point, attached to the end of the lever, and marking on metallic paper carried upon a revolving vertical cylinder, P. The distance between the two lines obtained by running the axle both ways, when measured on the above scale, indicated the value of the coefficient.

(2) *Method of Experimenting.*—Early in the experiments it was found, that immediately after the motion of the shaft was reversed, the friction was greater than it was when the shaft had been running in the same direction some time. This increase of friction, due to reversal, varied considerably. It was greatest with a new brass, and diminished as the brass became worn, so as to fit the journal more perfectly. Its greatest observed amount was at starting and was about twice the normal friction, and it gradually diminished until the normal friction was reached after about ten minutes continuous running. This increase of friction was accompanied by a strong tendency to heat and seize, even under a moderate load. In the case of one brass, which had worked for a considerable time without accident, and had consequently become worn so as to fit the journal very accurately, this tendency to increase of friction after reversal almost entirely disappeared; and it could be reversed under a full load without appreciable increase of friction or a tendency to heat or seize. The phenomenon must be due to the surface fibres of the metal, which have been for some time stroked in one direction, meeting point to point and interlocking when the motion is reversed. The very perfectly fitting brass was probably entirely separated from the journal by a film of oil; and there being no metallic contact the phenomenon did not show itself. In consequence of the above facts, it was found necessary to proceed with the experiments in the following order. A complete series of experiments, with a gradually increasing load, was taken with the journal running in one direction; the load was then diminished by the same steps as it had been increased, and the experiments thus repeated, the shaft still running in the same direction, until the load had thus

been reduced to 100 lbs. per square inch, which was the load due to the unweighted cradle. The direction of motion was then reversed, and the shaft run for half an hour, so as to get it thoroughly used to going the other way; after this the load could be increased and the experiments taken without any difficulty. The experiments, as before, were taken at each step whilst both increasing and decreasing the load; so that each recorded result is really the mean of four experiments, which have in many instances been taken several hours apart.

This method of obtaining a direct indication of the coefficient of friction, by the angular displacement of the frame connecting the brass and knife-edge, would undoubtedly have been the best had the coefficient of friction been nearly as constant as it has hitherto been supposed to be. But as shown by the results, the coefficient of friction was found, instead of being constant, to vary nearly inversely as the load, and also to be much smaller in quantity than was expected; the consequence was, that with high loads the height of the diagram was very small. In the cases where with the greatest loads, a coefficient of only $\frac{1}{1000}$ was observed, the distance between the two lines was only $\frac{1}{20}$ inch.

Owing to these experiments showing that the moment of friction was much more nearly constant than the coefficient, it was resolved to alter the method of observation, and to measure the moment directly, instead of the coefficient. For this purpose the paper cylinder was removed, and a small lever, M (see accompanying figure), was connected to the main indicating lever in such a manner that the motion of the end of the main lever was magnified five times at the end of the small lever. The end of the small lever was pointed; and when the machine was working, this point was brought exactly opposite a fixed mark by putting weights into a scale-pan on the end of the main lever. The main lever was so overbalanced that under all circumstances some weight was required to be added to the scale-pan, in order to bring the end of the small lever to the mark, even when, in addition to the friction being greatest, the direction of motion of the journal tended most to depress it. The method of running in both directions, and loading and unloading, was followed as before. The weights in the scale-pan being noted, the moment of friction was given by half the difference between the weights in the scale-pan, when running in one direction and in the other.



SECOND ARRANGEMENT
OF INDEX.

The following table is selected from those recorded in the *Proceedings of the Institution of Mechanical Engineers* as an example of the results obtained :—

BATH OF MINERAL OIL. TEMPERATURE 90° F. 4-INCH JOURNAL,
6 INCHES LONG. CHORD OF ARC OF CONTACT OF BRASS = 3·92 INCHES.

Nominal Load Lbs. per Sq. In.	COEFFICIENTS OF FRICTION, for speeds as below.						
	100 rev. 105 ft. per min.	150 rev. 157 ft. per min.	200 rev. 209 ft. per min.	250 rev. 262 ft. per min.	300 rev. 314 ft. per min.	350 rev. 366 ft. per min.	400 rev. 419 ft. per min.
625	...	·0013	·00139	·00147	·00157	·00165	...
520	...	·00123	·00139	·0015	·00161	·0017	·00178
415	...	·00123	·00143	·0016	·00176	·0019	·002
310	...	·00142	·0016	·00184	·00207	·00225	·00241
205	·00178	·00205	·00235	·00269	·00298	·00328	·0035
100	·00334	·00415	·00494	·00557	·0062	·00676	·0073

The above coefficients \times the nominal load = nominal frictional resistance per square inch of bearing.

Nominal Load Lbs. per Sq. In.	NOMINAL FRICTION RESISTANCE per square inch of bearing.						
	100 rev. 105 ft. per min.	150 rev. 157 ft. per min.	200 rev. 209 ft. per min.	250 rev. 262 ft. per min.	300 rev. 314 ft. per min.	350 rev. 366 ft. per min.	400 rev. 419 ft. per min.
	Lb.	Lb.	Lb.	Lb.	Lb.	Lb.	Lb.
625	...	·81	·865	·92	·98	1·03	...
520	...	·64	·72	·782	·84	·886	·924
415	...	·51	·594	·664	·73	·785	·83
310	...	·44	·494	·57	·64	·695	·745
205	·364	·419	·48	·55	·61	·67	·716
100	·334	·415	·494	·557	·62	·676	·73

N.B.—The bearing carried the 625 lbs. per sq. in. running both ways, but seized on the weight being increased.

The nominal load per sq. in. is the total load divided by 4×6 .

The actual load per sq. in. is the total load divided by $3·92 \times 6$.

These quantities were obtained by a direct load on the lever.

This was a thinner sample of mineral oil than that used in the previous experiments; it was fluid at 50°, while the oil previously used could only be described as grease at 50°. This will account for these experiments showing less friction than the former, except with the highest load, at which, the thin oil being overloaded and on the point of seizing, the friction is greater than with the thick oil.

Experiment showed that the friction varied considerably with temperature. All the oil-bath experiments were therefore taken at a nearly uniform temperature of 90° ; the variation above or below this temperature was never allowed to be more than $1\frac{1}{2}^{\circ}$.

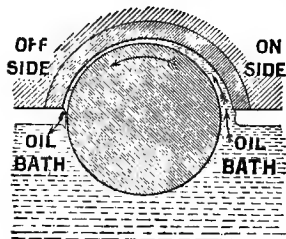
(3) *Results of Experiments.*—The results of the experiments are recorded in Tables I. to IX. in the *Proceedings of the Institution of Mechanical Engineers*. The general results of the oil-bath experiments may be described as follows:—*The absolute friction* (that is the actual tangential force per sq. in. of bearing, required to resist the tendency of the brass to go round with the journal) *is nearly a constant under all loads, within ordinary working limits*. Most certainly it does not increase in direct proportion to the load, as it should do according to the ordinary theory of solid friction. The ordinary theory of solid friction is, that it varies in direct proportion to the load; that it is independent of the extent of surface; and that it tends to diminish with an increase of velocity beyond a certain limit. The theory of liquid friction, on the other hand, is, that it is independent of the pressure per unit of surface, is directly dependent on the extent of surface, and increases as the square of the velocity. The results of these experiments seem to show that the friction of a perfectly lubricated journal follows the laws of liquid friction much more closely than those of solid friction. They show that under these circumstances the friction is nearly independent of the pressure per sq. in., and that it increases with the velocity, though at a rate not nearly so rapid as the square of the velocity.

The experiments on friction at different temperatures indicate a very great diminution in the friction as the temperature rises. Thus, in the case of lard oil, taking a speed of 450 revolutions per minute, the coefficient of friction at a temperature of 120° is only one-third of what it was at a temperature of 60° .

A very interesting discovery was made when the oil-bath experiments were on the point of completion. The experiments being carried on were those on mineral oil; and the bearing having seized with 625 lbs. per sq. in., the brass was taken out and examined, and the experiment repeated. While the brass was out, the opportunity was taken to drill a $\frac{1}{2}$ -in. hole for an ordinary lubricator through the cast-iron cap and the brass. On the machine being put together again and started with the oil in the bath, oil was observed to rise in the hole which had been drilled for the lubricator. The oil flowing over the top of the cap made a mess, and an attempt was made to plug up the hole, first with a cork and then with a wooden plug. When the machine was started the plug was slowly forced out by the oil in a way which showed that it was acted on by a considerable pressure. A pressure-gauge was screwed into the hole, and on

the machine being started the pressure, as indicated by the gauge, gradually rose to above 200 lbs. per sq. in. The gauge was only graduated up to 200 lbs., and the pointer went beyond the highest graduation. The mean load on the horizontal section of the journal was only 100 lbs. per sq. in. This experiment showed conclusively that the brass was actually floating on a film of oil, subject to a pressure due to the load. The pressure in the middle of the brass was thus more than double the mean pressure. No doubt if there had been a number of pressure-gauges connected to various parts of the brass, they would have shown that the pressure was highest in the middle, and diminished to nothing towards the edges of the brass.*

* Another set of experiments was afterwards made by Mr. Beauchamp Tower in order to investigate this point more thoroughly. The results formed the second report on Friction presented to the Institution of Mechanical Engineers in January, 1885. This report confirms the above statement. Small holes were bored in the brass bush, and a different one of these having been connected during each test with a Bourdon pressure gauge, and the bearing having then been immersed in an oil bath, the exact oil pressures at nine different points on the bearing were measured. The pressure was found to be greatest a little to the off side of the centre line of the bearing—i.e., to that side towards which the shaft turned, gradually falling to zero at each edge. It was also found to be greatest in the middle of the length of the bearing. The total upward force of these recorded pressures was found to be within a few pounds of the actual total load on the bearing, thus showing that the load was wholly supported by the film of oil which existed between the shaft and its brass bearing. Or, to quote Mr. Tower's own words, "It was possible to make the brass on a journal work so nicely that there was absolutely no metallic contact between the brass journal and the brass, the whole of the weight being borne by the oil. It seemed to him that the important practical inference was, that it was actually possible so to lubricate a bearing, that not only would metallic friction be altogether done away with, and thereby the amount of power lost by friction be reduced, but metallic wear and tear would also be done away with. He would not say that such a result was actually possible in practice now;



JOURNAL AND OIL BATH.

but it was a reasonable one to aim at in mechanism. By giving a sufficiently profuse lubrication, and by having the brasses so arranged, that there should be a uniform pressure all over their surface, it was possible to have wear and tear between metal and oil, instead of between metal and metal."

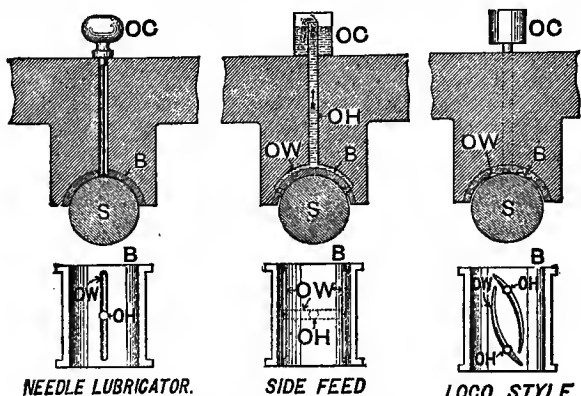
It is now generally recognised that the oil acts as a lubricant by merely furnishing molecules rolling in between the two surfaces, but unless these molecules can be got in, there is no possibility of the diminution of friction. The accompanying figure shows, in an exaggerated manner, what happens in the case of a well-lubricated journal when the brass is bored to a slightly larger radius than the shaft and the oil circulates freely, as shown by the small arrows.

The experiments with ordinary lubrication were begun with a needle lubricator, the hole from which penetrated to the centre of the brass. A groove in the middle of the brass, and parallel to the axis of the journal, extended nearly to the ends of the bearing for distributing the oil (see the first of the following three figures). It was found, that with this arrangement, the bearing would not run cool when loaded with only 100 lbs. per sq. in.; and that not a drop of oil would go down even when the needle-lubricator was removed and the hole filled completely with oil, thus giving a head of 7 inches of oil to force it into the brass. It appeared as though the hole and groove, being in the centre of pressure of the brass, allowed the supporting oil-film to escape. This view was confirmed by the following experiment:—The oil-hole being filled up to the top, the weight was eased off the journal for an instant. This allowed the oil to sink down in the hole and lubricate the journal; but immediately the load was again allowed to press on the journal the oil rose in the hole to its former level, and the journal became dry, thus showing that this arrangement of hole and groove, instead of being a means of lubricating the journal, was a most effectual one for collecting and removing all oil from it. It should be mentioned that care was taken to chamfer the edges of the groove, so as to prevent any scraping action between them and the journal.

As the centre of the brass was obviously the wrong place to introduce the oil, it was resolved to try to introduce it at the sides. Accordingly the centre hole and groove were filled up, and two grooves were made. These grooves were parallel to the axis of the journal, extending nearly to the ends of the brass, and were placed at equal distances on either side of the centre; they formed boundaries to an arc of contact, the chord of which was $3\frac{1}{4}$ inches (see the second of the following three figures). With this arrangement of groove the lubrication appeared to be satisfactory, the oil going down into the journal and the bearing running cool. The bearing nevertheless seized with an actual load of only 380 lbs. per square inch.

The arrangement of grooves was then altered to that usual in locomotive axle-boxes (see the third of the above three figures). The oil was introduced through two holes, one near each end of the brass, and each connected to a curved groove; the two curved grooves nearly enclosing an oval-shaped space in the centre of the brass. At the same time the arc of contact was reduced till its chord was only $2\frac{1}{4}$ inches. *This brass refused to take its oil or run cool.* It would sometimes run for a short time with an actual load of 178 lbs. per square inch, but rapidly

heated on the slightest increase of the load. The brass having been a good deal cut about by altering and filling up grooves, it was considered desirable to have a new brass, and one was accordingly obtained. *The grooves being made exactly the same as in the last experiment with the old one, this brass seized with an actual load of only about 200 lbs. per square inch.* The oil-box was completely cut away so as to allow a freer current of air round the bearing, and the lubricator pipes were soldered into the brass. The wicks were taken out of the lubricators and the



THREE METHODS OF LUBRICATING BEARINGS.

INDEX TO PARTS.

OC represents Oil cups.	OW represents Oil ways.
B „ Brass bearings.	S „ Shaft.
OH „ Oil holes.	

lubricators filled full of oil, by which means oil was supplied to the brass under a full head of 9 inches; and yet the oil refused to go down, and the underside of the journal felt perfectly dry to the hand, and speedily heated with a load of only 200 lbs. per square inch.

The fact that this arrangement of grooves, which is found to answer in the axles of railway vehicles, was found to be perfectly useless in this apparatus, can only be accounted for by the fact, that a railway axle has a continual end play while running, which prevents the brass from becoming the perfect oil-tight fit which it became in this apparatus. The attempts to make this arrangement of lubrication answer were not abandoned until after repeated trials. It now became clear that there was no use in trying to introduce the oil directly to the part of the

brass against which the pressure acted, and that the only way to proceed was to oil the lower side of the journal, and trust to the oil being carried round by the journal to the seat of the pressure.

The grooves and holes in the brass were accordingly filled up, and an oily pad, contained in a tin box full of rape oil, was placed under the journal, so that the journal rubbed against it in turning. The pad was only supplied with oil by capillary attraction from the oil in the box, and the supply of oil to the journal was thus very small; the oiliness in fact was only just perceptible to the touch, but it was evenly and uniformly distributed over the whole journal. The bearing fairly carried 551 lbs. per square inch, and three observations were obtained with 582 lbs., but the bearing was on the point of seizing and did seize after running a few minutes with this load. It will be observed that in this instance, the bearing seized with very nearly the same load as it did in the oil-bath experiment with rape oil.

These experiments with the oily pad show a nearer approach to the ordinarily received laws of solid friction than any of the others. The coefficient is approximately constant, and may be stated as about $\frac{1}{100}$ on an average. There does not in this case appear to be any well-defined variation of friction with variations of speed, according to any regular law.

The results of the experiments with rape oil, fed by a syphon lubricator to side grooves, follow nearly the same law as the results obtained from the oil-bath experiments, as far as the approximate constancy of the moment of friction is concerned; but the amount of the friction is about four times the amount in the oil-bath.

The oil-bath probably represents the most perfect lubrication possible, and the limit beyond which friction cannot be reduced by lubrication; and the experiments show that with speeds of from 100 to 200 feet per minute, by properly proportioning the bearing-surface to the load, it is possible to reduce the coefficient of friction as low as $\frac{1}{1000}$. A coefficient of $\frac{1}{500}$ is easily attainable, and probably is frequently attained in ordinary engine-bearings, in which the direction of the force is rapidly alternating and the oil given an opportunity to get between the surfaces, while the duration of the force in one direction is not sufficient to allow time for the oil-film to be squeezed out. The extent to which the friction depends on the quantity of the lubrication is shown in a remarkable manner by the following table, which proves that the lubrication can be so diminished that the friction is seven times greater than it was in the oil bath, and yet that the bearing will run without seizing:—

COMPARISON OF THE FRICTION WITH THE DIFFERENT METHODS OF LUBRICATION, UNDER AS NEARLY AS POSSIBLE THE SAME CIRCUMSTANCES. LUBRICANT RAPE OIL, SPEED 150 REVOLUTIONS PER MINUTE.

Mode of Lubrication.	Actual Load Lbs. per sq. in.	Coefficient of Friction.	Comparative Friction.
Oil-bath,	263	·00139	1
Syphon lubricator, .	252	·00980	7·06
Pad under journal, .	272	·00900	6·48

Observations on the behaviour of the apparatus gave reason to believe that with perfect lubrication the speed of minimum friction was from 100 to 150 feet per minute; and that this speed of minimum friction tended to be higher with an increase of load, and also with less perfect lubrication. By the speed of minimum friction is meant, that speed in approaching which, from rest, the friction diminishes, and above which the friction increases.

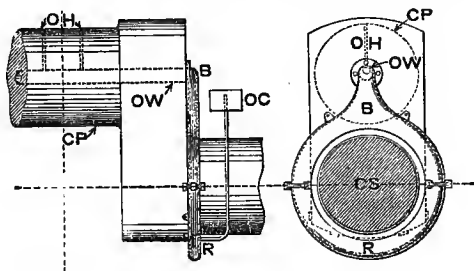
The following table gives the means of the actual frictional resistances at the surface of the journal per square inch of bearing, at a speed of 300 revolutions per minute, with all nominal loads from 100 lbs. per square inch up to 310 lbs. per square inch.

They also represent the relative thickness or body of the various oils, and (in their order, though perhaps not exactly in their numerical proportions) their relative weight-carrying power. Thus sperm oil, which has the highest lubricating power, has the least weight-carrying power; and though the best oil for light loads, would be inferior to the thicker oils if heavy pressures or high temperatures were to be encountered.

COMPARISON OF THE FRICTION WITH THE VARIOUS LUBRICANTS TRIED, UNDER AS NEARLY AS POSSIBLE THE SAME CIRCUMSTANCES. TEMPERATURE 90°, LUBRICATION BY OIL-BATH.

Lubricant.	Mean Resistance.	Per Cent.
	Lb.	
Sperm oil,	0·484	100
Rape oil,	0·512	106
Mineral oil,	0·623	129
Lard oil,	0·652	135
Olive oil,	0·654	135
Mineral grease, . .	1·048	217

Practical Examples of Lubricating Journals.—In the discussion which followed the reading of the foregoing report, a method of lubricating crank-pins was mentioned which has proved successful.



METHOD OF LUBRICATING CRANK PINS.

INDEX TO PARTS.

CS represents	Crank-shaft.	BR represents	Brass ring.
CP ,,	Crank-pin.	OW ,,	Oil way.
OC ,,	Oil cup.	OH ,,	Oil holes.

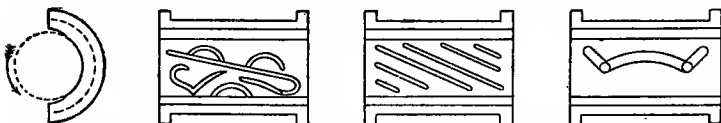
The oil from the oil cup, O C, passes into the hollow brass ring at R, and is driven outwards by centrifugal force to the point B, where it enters the oil way, O W. From thence it goes by the radial oil holes, O H, to form a film between the crank-pin, C P, and the surrounding brass bush of the connecting-rod end.

In discussing the third report, Mr. Daniel Adamson stated that his firm had adopted the method of cutting a flat on the shaft for the whole length of the journal of about $\frac{1}{8}$ inch wide for each inch in the diameter of the shaft up to 8-inch shafts and rather less for larger ones. In the case of heavy horizontal shafts, such as those supporting large flywheels, this method was found to effectually carry forward the oil into the bearing and thus produce smooth running.

Heavy Shaft Bearings.—The bearings illustrated on page 88 may serve their purpose fairly well where the pressure per square inch is small and continuous vibrations, shocks, and impacts (as in the case of a connecting-rod) are such as to facilitate the flow of the lubricant; but, when we have to deal with large, heavy, smooth running shafts, special care must be taken that the oil shall be naturally guided and spread by the direction of the rotation of the shaft.

The right-hand figure shows the plan of a top bush for a shaft revolving in one direction only, such as the crank shaft of a large central electric light or power station engine. Here the oil is led into the bearing from the oil box, and the curved

groove between the two oil holes, as well as the end lug-grooves, serve to spread the lubricant evenly over the bearing. The chamfering of the sides in this bush, as clearly shown by the end view, also helps to keep the journal equally cool all over. Should the shaft have to revolve in either direction, as in the case of a marine engine or a rolling mill, then similar oil grooves



OIL GROOVES FOR THE TOP AND LOWER HALVES OF HEAVY SHAFT BEARINGS.

should also be cut in the opposite direction to those shown in the figure, or they may be inclined as indicated by the next plan.

In the case of an outstanding end pedestal or plummer block bearing, such as that for supporting the outer end of a large dynamo, driven direct by the extension of the crank shaft of a steam engine, where very little vibration takes place, then the grooving may be cut on the front side of the bush, and take the form shown by the left-hand plan.* With forced lubrication the lower half of the bush is never chamfered, as it would allow of a drop in the pressure at which the oil is being forced into the bearing. As a practical example of the application of this peculiar grooving with carefully chamfered and neatly rounded edges, we may mention that of an end bearing of 30 inches diameter and 48 inches in length where the temperature was reduced from 120° F. to 98° F., or only 23° F. above that of the surrounding atmosphere.

Referring to the formulæ at the commencement of this Lecture, and substituting the actual values for this particular case, where $d = 30$ inches, $l = 48$ inches, $R = 60$ tons (as found by a hydraulic jack when lifting the shaft), we get the pressure per square inch of *nominal* bearing surface—

$$p = \frac{R}{dl} = \frac{60 \times 2,240}{30 \times 48} = 93.3 \text{ lbs.}$$

Also, when the shaft was revolving at its normal speed of 75 revolutions per minute, we get the sliding velocity—

$$v = \frac{\pi d n}{12} = \frac{22 \times 30 \times 75}{7 \times 12} = 590 \text{ feet per minute.}$$

* In the above cases the top bush only serves as a dust-protecting cap. I am indebted to Mr. J. L. Graham, Chief Engineer, for these practical hints on bearings, which he carried out upon one of the 5,000 H.P. engines at the Glasgow Tramway Power Station, Pinkston.

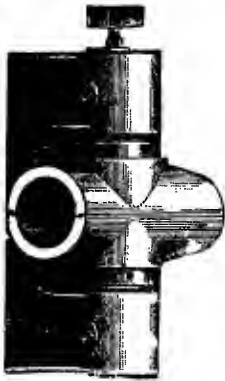
Eccentric Bearings.—Eccentrics should always be oiled from the top back side of their straps, and the latter should be turned slightly larger than the diameter of the eccentric pulley, in order that the oil may flow freely into the pulley and be carried round by it to the parts having the greatest pressure per square inch.*

Self-Oiling Bearings.†—The two following sets of figures show the method now commonly adopted for lubricating the end bearings of dynamo or electric motor armature shafts. The first one may be used for comparatively small machines up to 50 H.P., and the second for larger sizes up to 500 or more horse-power. The oil is poured in through the two top cap holes, C, and falls into an oil chamber or reservoir, O C, situated below the bush in the pedestal framing. Chains, rings, or loose collars, R, dip into this reservoir and automatically lift up a continuous stream of the lubricant, as they are rotated by contact with the shaft, S. The oil spreads to the right and left along the upper centre line of the shaft, and finds its way all over the same, until it reaches the end pockets, from whence it returns to the oil bath by the channels shown in the sectional side elevations. Oil drain taps, O T, are provided as shown for removing the oil when it becomes dirty, so that it may be filtered and again prepared for use. Some engineers place a felt pad in the centre of the upper half of the brass bearing (as shown by the first longitudinal section), with the view of straining the oil before it spreads from the loose rings, R, along the shaft. Usually, light brass dust caps, D C, are fitted to the outer ends of the bearings, and there should always be one or two sharp-edged oil throwers or spitters, s_1 , s_2 , turned upon the inner end of the shaft to prevent the oil creeping along the shaft and dirtying the commutator, or perhaps getting to and spoiling the insulation covering of the armature windings. These V-shaped spitters throw off the oil from their sharp edges by centrifugal force when the shaft is rotating at its normal speed. The oil in the oil chamber or reservoir should

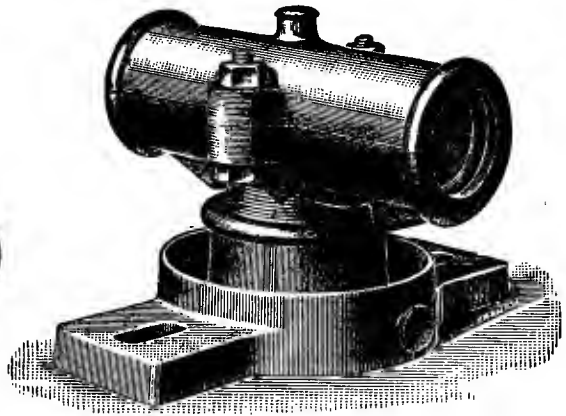
* Students are referred to the 14th and later editions of my *Text-Book on Steam and Steam Engines* for further facts and illustrations *re* lubrication of these bearings, engine crosshead slides, &c.

† The first pair of figures are from *The Treatise on the Theory and Practice of Lubrication*, by L. Archbutt and R. M. Deeley, as published by Charles Griffin & Co.; whilst the latter is from the 1902 Advanced Examination Paper on Machine Construction and Drawing, by the Board of Education. The former shows a short bearing of 6 to 8 inches, which can be automatically oiled by one loose collar, whereas, the latter is a bearing 1 foot 10 inches long and 8 inches diameter for a large alternate current dynamo, which requires two loose rings in order to efficiently lubricate such a length and size of bearing surface.

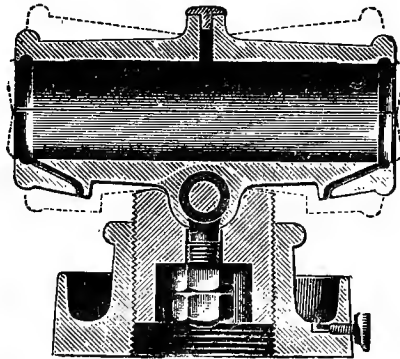
not be allowed to rise too high. Hence, in general practice an open-ended gauge glass is provided, so that the height of the oil may be carefully regulated.



BAGSHAW'S SWIVEL
ADJUSTABLE WALL-
BRACKET PEDESTAL.

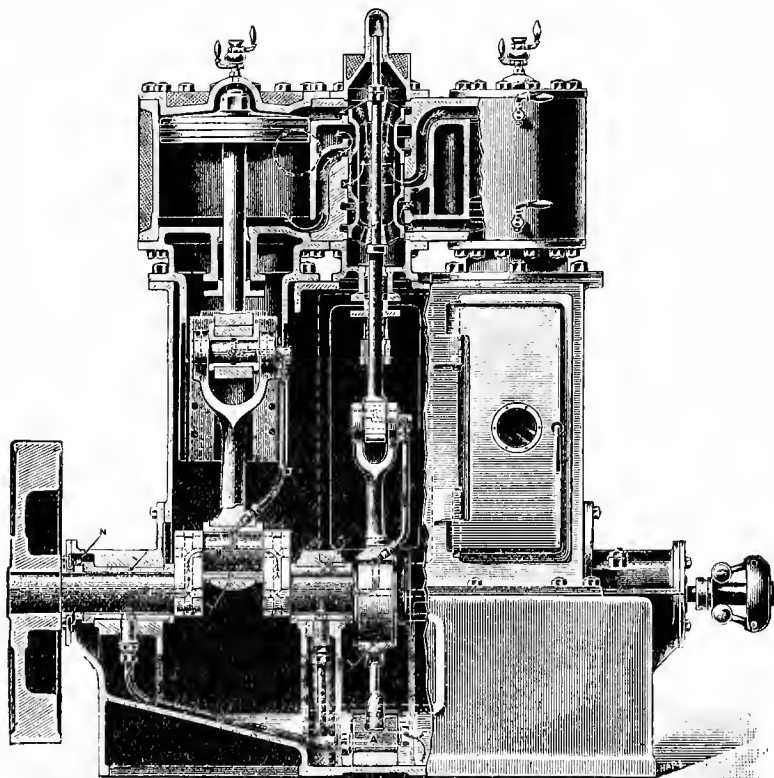


GENERAL VIEW OF ADJUSTABLE PEDESTAL.



VERTICAL SECTION OF ADJUSTABLE PEDESTAL.

Adjustable Bearings.—In many cases it is of very great importance to automatically (or by a combination of an automatic and other adjusting means) allow of the adjustment of a shaft (which may not be in perfect alignment or truth) to



BELLISS & COY.'S HIGH-SPEED COMPOUND ENGINE WITH CONTINUOUS FORCED LUBRICATION TO ALL BEARINGS EXCEPT CYLINDERS, VALVES, AND THEIR GLANDS.

lie close to its bearing without creating undue strains thereon. In the last figure under Self-Oiling Bearings we see that this is effected by a ball and socket arrangement, whereas in the three previous figures the adjustment for height is performed by means of a screw, and the sideways or swivel adjustment is performed automatically.

Automatic Engine Lubrication.—A good example of the lubrication of several journals and slide blocks from one common source of supply under pressure, is furnished by Belliss' high-speed compound engines for the direct driving of dynamos. It will be observed from the figure, that not only the main crank-shaft bearings, but also the crank-pins, slide-blocks, the upper ends of the connecting-rods, the piston-valve eccentric and its rods, are all supplied with oil from a small pump worked by the same eccentric which moves the piston valve. The oil is thereby forced through *each* bearing under a pressure of 10 lbs. per square inch, and is again and again sent on its soothing mission for months at a time, without change or great loss in quantity. A heavy lubricating oil is used, and it always returns to the small pump through a filter which removes any grit which it may have picked up from the bearings. This is a very different state of matters from the old "travelling oil-can" system, when the quantity of oil applied and the times of application were as erratic as the judgment of the attendant.

The Seigrist System of Automatic Lubrication for Large Engines.*—The complexity, size, and number of engines, dynamos, &c., which are now to be found in large central power stations—such as those in the Glasgow Electric Tramway Power House, at Pinkston, where over 22,000 H.P. is located in one room—have necessitated the discarding of the old system of oiling from dozens of isolated oil cups and hand-filled lubricators. Not only did the old system require a large number of men to carry the oil from the filtering plant to the engine room, but a considerable number had also to be constantly moving about, feeling bearings and pouring oil into the various cups, according to their respective requirements. The personal error or inattention of those so-called "travelling oil-cans," often entailed heated bearings, loss of power, and outlay for repairs, with waste of time and money for changing over, or the laying up of one set of engines until defects were overhauled.

* Students are referred to the Author's 14th and later editions of his *Text-Book on Steam and Steam Engines* for diagrams of the storage tanks, pumps, piping, valves, and filters required to successfully instal this system of automatic lubrication.

whenever the number of employees was diminished or the constant rigid system of supervision relaxed.

By the introduction of a properly installed and complete automatic system of lubrication, these shortcomings may be entirely overcome. Not only may all the valves, cylinders, slides, and bearings be thoroughly and efficiently oiled from one oiling table, but the number of attendants, their total wages bill, and the net quantity of oil used be very materially reduced.

In Pinkston Station, the oil in continuous circulation between drain pipes to filters, in filters and overhead tank would be equal to 500 gallons. The oil delivered on the bearings per hour was 50 gallons per engine. With the engine running twenty hours per day and seven weekly working days, the oil circulated on each engine per week is therefore 7,000 gallons. Now, with a weekly make up of 40 gallons of oil, that is equal to $\frac{4}{7}$ of a gallon lost per 100 gallons used by the machinery. No returns or pipes were used from the valve gear, pump room plant, or any auxiliary plant. The total savings in cylinder engine oil and waste, amounted to over £40 per week when compared with what it was before the new automatic system of lubrication was adopted.

Experiments on Collar Friction.—Mr. Tower also carried out some experiments on the friction of a collar bearing, for the purpose of ascertaining the friction in such cases as the thrust-bearings of propeller shafts. The results of these experiments constituted the third report of the Research Committee on Friction presented to the Institution of Mechanical Engineers in May, 1888. The collar or annular ring experimented with was made of mild steel, 12 inches inside and 14 inches outside diameter, and was pressed between two discs, the annular bearing surfaces of which were of gun-metal. Great difficulty was experienced with the lubrication, which was effected by means of four diametrical grooves cut in the face of the ring, $\frac{1}{4}$ of an inch wide and $\frac{3}{4}$ of an inch long. From each of these, there extended in the direction of motion a shallow serpentine groove $\frac{1}{16}$ of an inch wide and about $3\frac{1}{2}$ inches in length. These grooves were each supplied by a separate pipe into which oil was dropped from a reservoir. The minimum amount of lubrication necessary to prevent excessive heating of the bearing varied from 60 to 120 drops of mineral oil per minute, according to the pressure and velocity of the rubbing surfaces, but except with small pressures, it was found impossible to keep the bearing cool without water running over it. The pressure was varied from

15 to 90 lbs. per square inch, while the speeds ranged from 50 to 130 revolutions per minute. The results of these experiments seem to show, that (1) this kind of bearing is evidently much inferior to a cylindrical journal in its capability of carrying weight, in fact, 75 lbs. per square inch being the maximum that could be safely borne at the highest, and 90 lbs. per square inch at the lowest speed; (2) the friction in this case follows the law of solid friction much more nearly than that of liquids, or liquids and solids; (3) the coefficient of friction was independent of the speed but diminished slightly as the load was increased, and might be stated to be approximately $\cdot 05$ at 15 lbs. per square inch, diminishing to $\cdot 033$ at 75 lbs. per square inch. By far the most important factor, however, in determining the friction was the rate of lubrication, in fact, it was conclusively shown that the presence of friction meant non-lubrication. The following table shows the coefficient of friction at the different pressures and speeds:—

FRICTION OF A COLLAR BEARING.

(As Condensed by Prof. Unwin from Mr. Tower's Experiments)

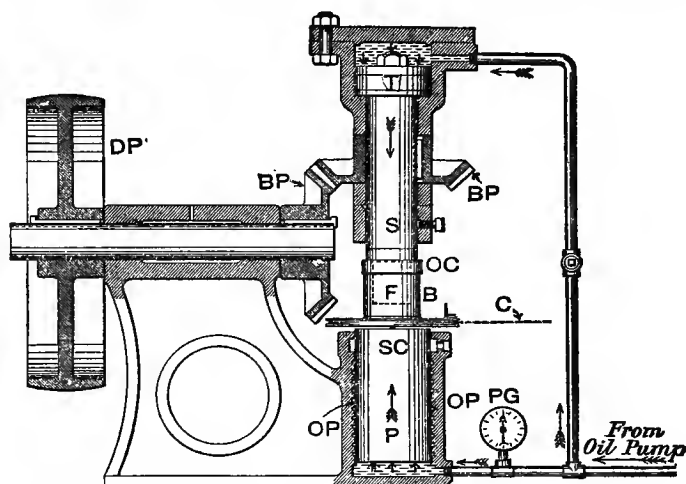
Intensity of Pressure, P, in lbs. per square inch.	SPEED IN REVOLUTIONS PER MINUTE.				
	50	70	90	110	130
15	$\mu = \cdot 045$	$\mu = \cdot 065$	$\mu = \cdot 043$	$\mu = \cdot 054$	$\mu = \cdot 064$
30	" $\cdot 037$	" $\cdot 048$	" $\cdot 050$	" $\cdot 049$	" $\cdot 048$
45	" $\cdot 036$	" $\cdot 040$	" $\cdot 036$	" $\cdot 036$	" $\cdot 037$
60	" $\cdot 029$	" $\cdot 038$	" $\cdot 036$	" $\cdot 037$	" $\cdot 041$
67	" $\cdot 035$	" $\cdot 033$	" $\cdot 035$	" $\cdot 036$	" $\cdot 038$
75	" $\cdot 035$	" $\cdot 034$	" $\cdot 035$	" $\cdot 035$	" $\cdot 036$
82	" $\cdot 034$	" $\cdot 032$	" $\cdot 035$
90	" $\cdot 031$	" $\cdot 044$

Mr. Thornycroft (the torpedo boat builder) said, that he limited the pressure on his thrust bearings to about 50 lbs. per square inch, and thus the limit of 70 to 80 lbs. arrived at by these experiments, received confirmation from his extensive practical experience of similar collar bearings. The pressures which can, however, be thus carried, depend (1) on the hardness and truth of the rubbing surfaces.* (2) On the freedom with

* Thus, hardened steel working in a dense cast-iron bearing when well lubricated is capable of withstanding a greater pressure per square inch than anything else.

which the lubricant can get in between the rubbing surfaces. This is often assisted by dithering or trembling or alternate pressure and relief, such as takes place at the thrust block of a steamer. (3) On the facilities for dissipating the heat generated through friction by admitting air freely to the bearing, since the rate at which heat was generated constituted the true limit to the load which a bearing will carry.

Friction of a Pivot Bearing.—The experiments on this kind of bearing formed the fourth report of the Research Committee, which was presented to the Institution of Mechanical Engineers in March, 1891. From the two following figures with index



APPARATUS FOR ASCERTAINING THE FRICTION OF A PIVOT BEARING.

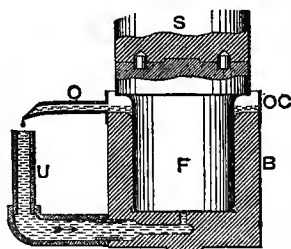
INDEX TO PARTS.

F	represents	Footstep.	PG	represents	Pressure Gauge.
B	"	Bearing.	OC	"	Oil Chamber.
OP	"	Oil Press.	S	"	Upright Shaft.
P	"	Plunger.	T	"	Top of Shaft.
SC	"	Steel Centre.	BP	"	Bevel Pinions.
L	"	Lever Plate.	DP	"	Driving Pulley.
C	"	Chain.			

to parts, and the following abbreviated description, the student will have no difficulty in comprehending how these important experiments were carried out. The footstep, F, and its bearing, B, were flat-ended and of 3 inches diameter. They were pressed

together with a known force by the aid of a small hand oil pump. The oil from this pump (which was fitted with an air-vessel) passed below the plunger, P, of the oil press, O P, and, at the same time, it acted upon the top, T, of the vertical shaft, S, to the lower end of which the footstep was fixed in the manner shown by the smaller figure. The pressure of the oil thus supplied from the pump was indicated by the pressure gauge, P G. Into the top of the plunger, P, there was inserted a piece of hard steel having a conical centre or centre-pop, wherein rested a hard steel centre, S C, screwed into the under side of the lever plate, L, which carried the bearing, B.

A small chain, C, was fastened to this circular plate, L, and lay in the groove turned in its periphery. The other end of the chain was so connected to a spring-balance (not shown) that any tendency of the plate to rotate (due to the friction between the



METHOD OF LUBRICATING FOOTSTEP AND BEARING.

INDEX TO PARTS.

S	represents	Shaft.	U	represents	Upright Oil Pipe.
F	„	Footstep.	OC	„	Oil Chamber.
B	„	Bearing.	O	„	Overflow Pipe.

footstep and the bearing), stretched the balance and thereby the frictional moment between the footstep and its bearing was measured in inch-pounds. The upright shaft, S, received motion through the two bevel pinions, B P, a horizontal shaft and the driving pulley, D P, which was connected by a belt to a suitable motor. The lubrication of the footstep and its bearing was carried out automatically; for the arrangement shown in the annexed figure acted like an oil pump. The mineral oil from the pipe, U, passed freely by gravity to the centre of the footstep, then radially along a diametrical groove, spirally over the flat surface, and finally it was forced up the sides of the bearing to the oil chamber, O C, from which it again passed to the pipe,

U, by the overflow, O. In fact, the faster the speed of rotation the quicker was the circulation of the oil.*

Results of the Experiments.—A series of experiments was first made with a steel footstep on a manganese bronze bearing, at speeds of 50, 128, 194, 290, and 353 revolutions per minute with loads varying from 20 to 160 lbs. per square inch of the flat surface. The manganese bronze bearing was then replaced by one with a white metal bearing surface, and observations of the friction at the various loads were made at 128 revolutions per minute. The coefficient of friction was obtained by dividing the readings of the spring balance as ascertained in inch-pounds by the total load on the bearing, or from the formula:—

$$\mu = \frac{S \times L}{P \times A}.$$

Where S = spring-balance reading in pounds.

„ L = leverage of chain.†

„ P = pressure on the gauge shown in pounds per square inch.

„ A = area of bearing in square inches.

From the results thus obtained, it was found that the coefficient of friction was slightly larger with the white metal than with the manganese bronze, but the difference was so small that the results may be looked upon as identical. Since the friction was mainly between *oil* and metal, instead of between *metal* and metal, it should be independent of the nature of the metal. Hence, it may be urged that with a perfect system of lubrication, applied under pressure to a bearing by means of a force pump, it should not matter much of what material the bearing and shaft or pivot are composed, so long as they are perfectly smooth and true. An examination of the results (see the accompanying set of curves) also show that the higher the speed the less the coefficient of friction became—*e.g.*, from .0196 with a load of 20 lbs. per square inch at 50 revolutions per minute it fell to .0167

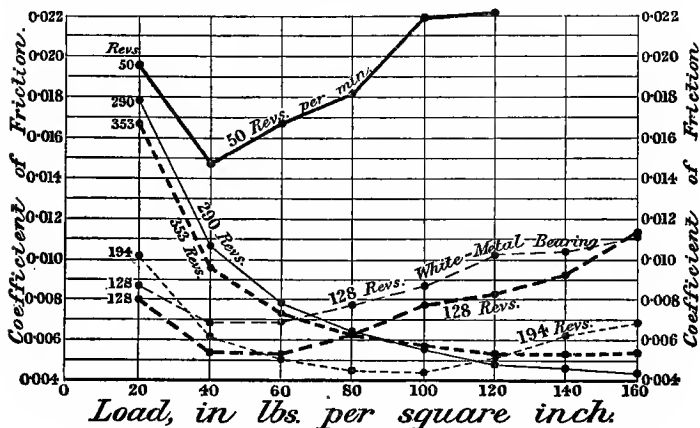
* It is worthy of mention that two opposite radial grooves were found to act better than three or four or any other number.

† The true leverage of the chain is the actual leverage divided by the distance from the centre of the shaft to the centre of frictional resistance, which was assumed to be in this case 1 inch from the centre of the shaft. The centre of frictional resistance being assumed to be 1 inch from the centre of the shaft, we have:—

$$F \times 1 = S \times L$$

i.e., the moment of frictional resistance = $S \times L$.

at 353 revolutions per minute with the same load; and from $\cdot 0221$ at 50 revolutions and 120 lbs. load to $\cdot 0054$ at 353 revolutions and the latter load. Unfortunately, we think it was not proved how much of this reduction of the coefficient was due to increase of speed per second; or whether the lower coefficient could not have been got at the lower speeds if equally good lubrication had been maintained. Some of this reduction at least (in the



CURVES SHOWING RESULTS OF EXPERIMENTS ON PIVOT FRICTION.*

absence of direct observation on the point) may be put down to the better lubrication which the bearing or pivot automatically received at the higher speeds. Also, in actual practice, whenever dithering or trembling comes into play, the lubricant gets more readily between the surfaces and thus produces more thoroughly the effect of liquid friction.

In the discussion which took place on the above reports, it was pointed out that the introduction of two or more loose, hardened, steel washers between the bottom of the footstep and its bearing, or between a collar and its bearing, enabled heavier loads to be carried than without them. A difference of opinion was expressed as to their precise action. We think, however, that Mr. Tower's explanation was the best, viz.:—that only one pair of

* It would have been better, if after marking the points showing the several observations, smooth curves had been drawn through the mean positions. We have, however, reproduced the above figure direct from the *Proceedings of the Institution of Mechanical Engineers*.

these interposed discs were rubbing at one time, but when these became heated the smallest tendency to seize occasioned more friction between the working pair than between some other pair; consequently, these latter took up the work and thus the work was alternately divided between the several pairs of discs, giving time for each pair to cool before they again come into action.

Experiments on the Friction of Railway Brakes.*—In 1878, Captain Douglas Galton and Mr. George Westinghouse carried out some careful experiments on the friction of railway brakes. The brake blocks were made of cast-iron and the wheels had steel tyres. The pressure, and also the friction, between the brake blocks and peripheries of the wheels were automatically recorded by means of hydraulic gauges. Two series of experiments were made; the first, to determine the coefficient of friction between the brake blocks and the tyres, which we shall term the "brake coefficient"; and the second, to determine the coefficient of friction between the wheels and the rails, when the former were "skidded," or prevented from rotating, which we shall term the "rail coefficient." From these experiments, the brake coefficient was generally greater with low than with high speeds. Thus, immediately after the application of the brakes, the brake coefficient was 0.18 for a speed of 17 miles per hour, while at a speed of $47\frac{1}{2}$ miles per hour, the coefficient was only 0.132. After the brakes had been on for 5 seconds the coefficients at these speeds were 0.157 and 0.07 respectively. When the brakes had been on for 15 seconds, the coefficients were further reduced to 0.11 and 0.055 respectively. Thus we see, that the brake coefficient not only diminished as the speed increased, but diminished the longer the brake had been in contact with the wheel. As the speed decreased, the friction between the wheel and the brake continued to increase, until it became equal to the friction between the wheel and the rail. Then the wheel ceased to rotate and skidded along the rail. The "rail coefficient" was much lower than the "brake coefficient" and increased as the speed decreased. This increase was slow at first but increased greatly as the speed got less, until, when the carriage was about to stop, or just before skidding, it became even greater than the "brake coefficient." The rail coefficient was also found to be greater with steel tyres on iron rails at high speeds, than with steel tyres on steel rails.†

* See *Proc. Inst. M.E.*, June and October, 1878, and April, 1879.

† In *The Practical Engineer* for July 20, 1894, p. 49, it is stated that soft forged steel brake shoes have been proved to last much longer, wear the wheels less, and to be quite as effective as cast-iron.

Friction between Water and Bodies Moving through it.*—The frictional resistance between water and bodies passing through it has been investigated by Col. Beaufoy in a memorable series of experiments carried out in the Greenland Dock, near London, early in this century, and more recently by Dr. William Froude at Chelston Cross, and by Dr. Tideman at Amsterdam.

Dr. Froude's experiments, being the most thorough and conclusive, and those most commonly referred to in the calculations involved in the resistance of ships, we shall briefly describe them, as well as give a few of his results.

They were made in a still water tank 278 feet long, 36 feet wide, and 8 feet 9 inches deep.

The surfaces experimented upon were wooden planks, $\frac{3}{16}$ inch thick, varying from 1 foot to 50 feet long, having both bow and stern sharpened so as to eliminate resistances other than frictional. These planks (with their upper edges $1\frac{1}{2}$ inches below the water level) were suspended from a carefully balanced framework (free to swing without friction fore and aft) attached to a dynamometric truck (set in motion by an endless steel rope) on rails fixed over the tank and running its entire length. The resistance of the specimen was communicated through a spiral spring to a lever, actuating a pen which recorded the tension on a cylinder revolving synchronously with the wheels of the truck. The time was separately registered by means of a pen connected with a chronometer.

Dr. Froude set himself to determine :—

1. The law of the variation of the resistance in terms of the velocity.
2. The law of the variation of the resistance in terms of the length of surface.
3. Variations of resistance with varying qualities of surface.

The second of these problems requires a word of explanation. Whereas, with short lengths, the resistance varied sensibly as the squares of the velocities, the rate of variation was found to fall continuously as the lengths were increased, until it became as low as the 1.83 power of the velocities with a specimen of 50 feet, which was the greatest length experimented upon. Dr. Froude has pointed out, that the cause of this diminution is to be sought in the effect of the forward motion, imparted by the friction of the surface to the stream lines in contact with it, or nearest to it. These stream lines have consequently a lower velocity, relatively

* This latter part of Lecture VI. was kindly contributed by Dr. Robert Caird, of Messrs. Caird & Co., Shipbuilders and Engineers, Greenock.

TABLE I.

SURFACE.	LENGTH OF THE SURFACE OR DISTANCE FROM CUTWATER.											
	2 Feet.			8 feet.			20 feet.			50 feet.		
	A.	B.	C.	A.	B.	C.	A.	B.	C.	A.	B.	C.
Varnish, . . .	2.00	0.41	0.390	1.85	0.325	0.264	1.85	0.278	0.240	1.83	0.250	0.226
Paraffin, . . .	1.95	0.38	0.370	1.94	0.314	0.260	1.93	0.271	0.237
Tinfoil, . . .	2.16	0.30	0.295	1.99	0.278	0.263	1.90	0.262	0.241	1.83	0.246	0.232
Calico, . . .	1.93	0.87	0.725	1.92	0.626	0.504	1.89	0.531	0.447	1.87	0.474	0.423
Fine Sand, . . .	2.00	0.81	0.690	2.00	0.583	0.450	2.00	0.480	0.384	2.06	0.405	0.337
Medium Sand, . . .	2.00	0.90	0.730	2.00	0.625	0.488	2.00	0.534	0.465	2.00	0.488	0.456
Coarse Sand, . . .	2.00	1.10	0.880	2.00	0.714	0.520	2.00	0.588	0.490

to the specimen being towed past them, than the undisturbed water.

The law of variation in terms of length of surface has not yet been satisfactorily investigated for lengths beyond 50 feet, and further experiments are very desirable.

The foregoing table sums up the results of these frictional experiments. The columns A, B, and C respectively refer to:—

A. The power of the velocity, to which the resistance is sensibly proportional.

B. Resistance in pounds per square foot of surface taken as a mean resistance over the whole length.

C. Resistance per square foot taken at the specified distances abaft the cut water which are given at the head of the columns.

The resistances in this Table are those due to a velocity of 600 feet per minute.

The following table is deduced from these experiments and is in a form suitable for the Naval Architect:—

TABLE II.

FROUDE'S FRICTIONAL CONSTANTS FOR SALT WATER, PARAFFIN OR SMOOTHLY PAINTED SURFACES.

Length of Vessel or Model in Feet.	Coefficient of Friction. For Speed in Knots.	Speed Power according to which Friction Varies.	Length of Vessel or Model in Feet.	Coefficient of Friction. For Speed in Knots.	Speed Power according to which Friction Varies.
	μ	n		μ	n
8	·01197	1·825	80	·00933	1·825
9	·01177	„	90	·00928	„
10	·01161	„	100	·00923	„
12	·01131	„	120	·00916	„
14	·01106	„	140	·00911	„
16	·01086	„	160	·00907	„
18	·01069	„	180	·00904	„
20	·01055	„	200	·00902	„
25	·01029	„	250	·00897	„
30	·01010	„	300	·00892	„
35	·00993	„	350	·00889	„
40	·00981	„	400	·00886	„
45	·00971	„	450	·00883	„
50	·00963	„	500	·00880	„
60	·00950	„	550	·00877	„
70	·00940	„	600	·00874	„

Frictional Resistance of a Ship Propelled through Sea Water.—The frictional resistance of a ship is readily calculated from Table II. by the formula :—

$$R = \mu S V^n$$

Where, R = Resistance in pounds.

μ = Coefficient of friction.

S = Wetted surface in square feet.

V = Velocity in knots.

n = The power according to which friction varies.

The total resistance of a ship when propelled through water is composed of :—

1. Frictional resistance.
2. Eddy-making resistance.
3. Wave-making resistance.

The total resistance of a model is measured by a dynamometric apparatus identical with that described above. From it, the frictional resistance calculated from the table is deducted and the balance is the eddy-making and wave-making resistance (or residuary resistance). The calculation of the residuary resistance of an actual ship from that of a model follows what is called Froude's Law of Comparison, which, briefly stated, is :—

If the linear dimensions of a ship are λ times those of its model, and if, at the velocities $v_1, v_2, v_3 \dots$ of the model in water, the resistances are $r_1, r_2, r_3 \dots$ then the resistances $R_1, R_2, R_3 \dots$ of the ship, at the velocities $V_1, V_2, V_3 \dots$ (which are respectively equal to $v_1\sqrt{\lambda}; v_2\sqrt{\lambda}; v_3\sqrt{\lambda} \dots$) will be $R_1 = \lambda^3 r_1; R_2 = \lambda^3 r_2; R_3 = \lambda^3 r_3 \dots$

EXAMPLE I.—Applying the foregoing to the following case :—

Model 10 ft. long ; 1.194 ft. broad ; 0.555 ft. draught of water.

Ship 360 " ; 43 " ; 20 " "

$$\lambda = 36 ; \sqrt{\lambda} = 6 ; \lambda^3 = 46,656.$$

Calculation of Frictional Resistance :—

$$\mu = .00888 \text{ (from Table II.)}$$

$$S = 24,500 \text{ square feet.}$$

$$V = 12 \text{ knots.}$$

$$V^{1.825} = 93.219.$$

$$R = \mu S V^{1.825} = 20,280.7 \text{ lbs.} = 9.053 \text{ tons.}$$

Residuary Ship Resistance.—If at a velocity of 3·378 feet per second (equal to 2 knots) the tank trial of the model gives a residuary resistance $r = \cdot 23$ lb., the corresponding speed of the ship will be:—

$$v \sqrt{\lambda} = 3 \cdot 378 \times 6 = 20 \cdot 268 \text{ feet per second.}$$

$$\text{„ „} = 1216 \text{ feet per minute.}$$

$$\text{„ „} = 12 \text{ knots.}$$

And the residuary resistance of the ship at that velocity will be:—

$$\lambda^3 r = 46,656 \times \cdot 23 = 10,731 \text{ lbs.} = 4 \cdot 79 \text{ tons.}$$

Total Ship Resistance:—

$$\text{Frictional resistance (as above)} = 9 \cdot 053 \text{ tons.}$$

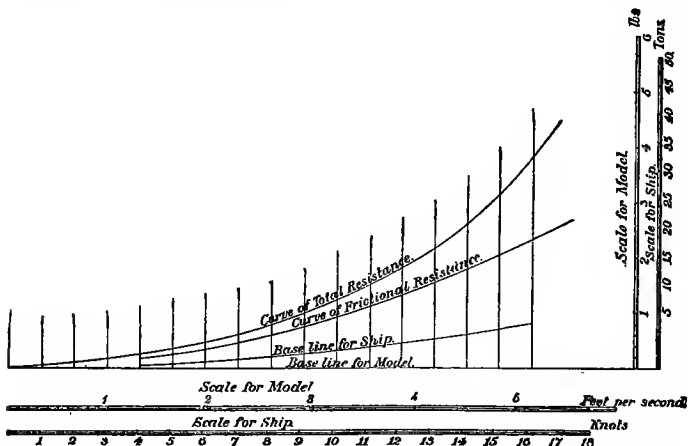
$$\text{Residuary resistance („)} = 4 \cdot 790 \text{ „}$$

$$\text{Total resistance} = \underline{\underline{13 \cdot 843 \text{ „}}}$$

Reducing this to horse-power it becomes:—

$$\frac{13 \cdot 85 \times 2240 \times 1216}{33,000} = 1142 \cdot 6 \text{ Effective Horse-power.}$$

$$\left. \begin{array}{l} \text{And, with a propulsive co-} \\ \text{efficient of } \cdot 5 \text{ or } 50 \text{ per cent.} \end{array} \right\} = 2285 \cdot 2 \text{ Indicated Horse-power.}$$



TOTAL AND FRICTIONAL RESISTANCE OF A SHIP PROPELLED THROUGH SEA WATER.

The calculation may, otherwise, be made directly from the total resistance of the model, correcting for surface friction by the method elaborated by Mr. R. E. Froude in his paper read before the Institute of Naval Architects in 1888.

The above diagram shows the results of a similar calculation in graphic form. This figure is reproduced from an actual diagram worked out from a tank trial where λ (or the linear ratio of dimensions between ship and model) was equal to 28.

Here, 1 knot for the model = $\sqrt{28} = 5.2915$ knots for ship.

Or, 3 knots for model = $3\sqrt{28} = 15.8745$, or $15\frac{7}{8}$ knots for ship.

EXAMPLE II.—Deduce from the data employed in the previous Example I, the total resistance of the model at the specified speed.

ANSWER :—

$$\left. \begin{array}{l} \text{Wetted surface of} \\ \text{model, } s \end{array} \right\} = \frac{S}{\lambda^2} = \frac{24,500}{36^2} = 18.9 \text{ square feet.}$$

$$\left. \begin{array}{l} \text{Friction coefficient} \\ \text{for model, } \mu \end{array} \right\} = 0.01161 \text{ (Table II.).}$$

$$\text{Speed of model, } v = \frac{V}{\sqrt{\lambda}} = 2.0 \text{ knots.}$$

$$\begin{aligned} \left. \begin{array}{l} \text{Frictional resistance} \\ \text{of model, } R_m \end{array} \right\} &= \mu \times s \times v^{1.825} \\ \text{,, ,,} &= 0.01161 \times 18.9 \times 2^{1.825} \\ \text{,, ,,} &= 0.01161 \times 18.9 \times 3.543 \\ \text{,, ,,} &= 0.777 \text{ lb.} \end{aligned}$$

Hence,

$$\begin{aligned} \left. \begin{array}{l} \text{Total resistance} \\ \text{of model} \end{array} \right\} &= \text{Frictional resistance} + \text{Residuary resistance} \\ &= R_m + r_m = 0.777 + 0.230 = 1.007 \text{ lbs.} \end{aligned}$$

EXAMPLE III.—Suppose a shipbuilder obtained a contract to build a vessel 400 feet long, 50 feet broad, and 25 feet draught, how would he determine, by aid of a proportionately sized model, the effective horse-power required to propel the ship at 15 knots?

ANSWER.—He would make a model of the vessel, to, say, a

scale of $\frac{3}{8}$ inch to the foot, and tow the same in a tank at the speed given by the ratio, $V/\sqrt{\lambda}$.

Preliminary Calculations for Dimensions of Model.—

$$\text{Here, } \lambda = \frac{\text{Length of vessel}}{\text{Length of model}} = \frac{12}{\frac{3}{8}} = 32.$$

$$\text{Hence, Length of model} = \frac{400}{32} = 12.5 \text{ feet.}$$

$$\text{Breadth of model} = \frac{50}{32} = 1.56 \text{ feet.}$$

$$\text{Draught of model} = \frac{25}{32} = 0.78 \text{ feet.}$$

$$\text{Speed of model, } v = \frac{V}{\sqrt{\lambda}} = \frac{15}{\sqrt{32}} = 2.651 \text{ knots.}$$

Now, from actual trials of a model having these dimensions, it was found that the total pull, $P = 2.85$ lbs. for the speed 2.651 knots (or nearly 4.475 feet per second), and that the wetted surface, $s = 29.3$ square feet. Taking the coefficient of friction for model as $\mu = 0.01125$ (from Table II. of this book) by interpolation, we get:—

$$\begin{aligned} \left. \begin{array}{l} \text{Frictional resistance} \\ \text{of model, } R_m \end{array} \right\} &= \mu \times s \times v^{1.825} \\ \text{,,} \quad \text{,,} &= 0.01125 \times 29.3 \times 2.651^{1.825} \\ \text{,,} \quad \text{,,} &= 0.01125 \times 29.3 \times 5.9255 \\ \text{,,} \quad \text{,,} &= 1.95 \text{ lbs.} \end{aligned}$$

$$\left. \begin{array}{l} \text{Residuary resistance} \\ \text{of model, } r_m \end{array} \right\} = P - R_m = 2.85 - 1.95 = 0.90 \text{ lb.}$$

Now, with the data in this form, we are in a position to estimate the total resistance and effective horse-power required for driving the full-sized ship at the given speed of 15 knots.

$$\left. \begin{array}{l} \text{Wetted surface} \\ \text{of ship, } S \end{array} \right\} = \lambda^2 \times s = 32^2 \times 29.3 = 30,000 \text{ sq. ft. (say).}$$

$$\left. \begin{array}{l} \text{Ship's coefficient} \\ \text{of skin friction, } \mu \end{array} \right\} = 0.00886 \text{ (by Table II.).}$$

$$\begin{aligned}
 \therefore \quad \left. \begin{array}{l} \text{Frictional res. of} \\ \text{ship, } R \end{array} \right\} &= \mu \times S \times V^{1.825} \\
 &= 0.00886 \times 30,000 \times 15^{1.825} \\
 &= 0.00886 \times 30,000 \times 140.07 \\
 &= 37,230 \text{ lbs.}
 \end{aligned}$$

$$\text{And, } \left. \begin{array}{l} \text{Residuary res.} \\ \text{of ship, } r \end{array} \right\} = \lambda^3 \times r_m = 32^3 \times 0.90 = 29,491 \text{ lbs.}$$

$$\text{Hence, Total res. of ship} \left. \begin{array}{l} \text{at 15 knots} \end{array} \right\} = R + r = 37,230 + 29,491 = 66,721 \text{ lbs.}$$

$$\therefore \text{ E.H.P.} = \frac{66,721 \text{ lbs.} \times \left(\frac{6,080 \times 15}{60} \right) \text{ feet per minute}}{33,000}$$

$$\therefore = \frac{66,721 \times 1,520}{33,000} = 3,073 \text{ Effective Horse-power.}$$

But, this is merely the horse-power that would be expended in towing a ship of these dimensions at 15 knots; whereas, the marine engineer has to provide an engine, shafting and screw, &c., capable of delivering 3,073 effective horse-power to the direct propulsion of the vessel. Consequently, he must multiply the same by a coefficient found from actual practice, to make up for friction losses throughout every part of his propelling machinery, and for slip as well as friction waste of the screw propeller itself. It has been found that this coefficient is seldom less than 2, and may even be more. Since, frequently, 40 to 50 per cent. of the I.H.P. (as found by indicator diagrams) can only be accounted for in moving a vessel of a certain displacement tonnage at a certain rate. Taking the coefficient $\frac{\text{I.H.P.}}{\text{E.H.P.}} = \frac{2}{1}$ in this case.—

We get, the I.H.P. = 2 E.H.P. = $2 \times 3,073 = 6,146$.

Note.—This corresponds to about two-thirds of a horse-power per ton, for a vessel of the above dimensions.

EXAMPLE IV.—If, however, in Example III., the mechanical efficiency of the propelling engines be estimated at 86 per cent., and the screw propellers be designed to give an efficiency of 65 per cent.; then, what horse-power must the engines indicate, and what will be the propulsive efficiency?

Also, calculate the power expended on:—

- (a) Engine friction, &c.
- (b) Screw friction and slip.
- (c) Skin friction resistance.
- (d) Residuary resistance.

ANSWER.—

$$\left. \begin{array}{l} \text{H P. transmitted by} \\ \text{engines to propellers} \end{array} \right\} = 0.86 \times \text{I.H.P.} = \text{B.H.P.} \dots \text{(I)}$$

$$\left. \begin{array}{l} \text{H.P. transmitted by} \\ \text{propellers to ship} \end{array} \right\} = 0.65 \times \text{B.H.P.}$$

$$= 0.65 \times 0.86 \times \text{I.H.P.}$$

$$= 0.559 \times \text{I.H.P.} = \text{E.H.P.} \dots \text{(II)}$$

$$\therefore \text{I.H.P.} = \frac{\text{E.H.P.}}{0.559} = \frac{3,073}{0.559} = 5,497.$$

$$\left. \begin{array}{l} \text{The propulsive} \\ \text{efficiency} \end{array} \right\} = \frac{\text{E.H.P.}}{\text{I.H.P.}} = 0.559 = 55.9 \text{ per cent.}$$

Also, by Equation I. we have:—

$$\text{B.H.P.} = 0.86 \times \text{I.H.P.}$$

$$\text{Or, B.H.P.} = 0.86 \times 5,497 = 4,727.$$

$$\begin{array}{ll} \text{(a) H.P. expended on engine friction} & \text{Percent.} \\ & \text{I.H.P.} \\ & = \text{I.H.P.} - \text{B.H.P.} = 5,497 - 4,727 = 770 = 14.0 \end{array}$$

$$\begin{array}{ll} \text{(b) H.P. expended on screw friction and slip} & \\ & = \text{B.H.P.} - \text{E.H.P.} = 4,727 - 3,073 = 1,654 = 30.1 \end{array}$$

$$\begin{array}{ll} \text{(c) H.P. expended on skin friction} & \\ \text{R} \times \text{Speed of ship in ft. per min.} & 37,230 \times 1,520 \\ = \frac{\text{One H.P. in ft.-lbs. per min.}}{33,000} & = \frac{1,715}{33,000} = 1,715 = 31.2 \end{array}$$

$$\begin{array}{ll} \text{(d) H.P. expended on residuary resistance} & \\ r \times \text{Speed of ship in ft. per min.} & 29,491 \times 1,520 \\ = \frac{\text{One H.P. in ft.-lbs. per min.}}{33,000} & = \frac{1,358}{33,000} = 1,358 = 24.7 \end{array}$$

$$\text{Total I.H.P.} = \underline{\underline{5,497}} = \underline{\underline{100.0}}$$

Note.—This is the result which we should expect to get from trials on the measured mile, with a ship having moderately fine lines, newly cleaned and painted bottom, and with her propelling machinery working in first-class order.

LECTURE VI.—QUESTIONS.

1. Give a concise account of General Morin's, Hirn's, and Thurston's experiments on friction, and explain wherein their conclusions fell short of the results arrived at by the Institution of Mechanical Engineers.

2. Give an account of some experiments on the friction of a well lubricated journal, and state what has been ascertained as to the magnitude of the friction under varying loads, temperature, and velocity. Also state what you know as to the intensity of the pressure at different points of the bearing surface.

3. Give the results of some experiments which have been made to determine the coefficient of friction in a well lubricated bearing, and the greatest pressure to which the bearing may be subjected. State also at what portion of the surface of the bearing the lubricant should be introduced. Describe and give a sketch of the construction of a bearing for a shaft which is required to run at a very high speed, assuming that the revolving parts cannot be perfectly balanced.

4. Describe and sketch any form of machine for measuring the friction of lubricated journals. Show how you would deduce from it the coefficient of friction.

5. Give a short account of Mr. Beauchamp Tower's experiments and results on the friction of collar bearings.

6. State what you know about the friction of pivot bearings. Sketch and describe any apparatus which has been used for determining the same.

7. What is your idea of the most perfect system of lubricating engine bearings? Give reasons for your answer, with sketches.

8. State what you know about the friction between railway carriage wheels and their brakes and the permanent way rails.

9. How is the frictional resistance between a moving ship and sea water determined? What is meant by residuary and total ship resistance?

10. Describe, without too much mathematics, the explanation given by Professor O. Reynolds of Mr. Beauchamp Tower's experiments on journal friction.

11. How do tank experiments give information as to the indicated power of the engines of a ship which is being designed? (S. & A. Hons. Exam., Part II., 1898.)

12. Taking advantage of the data for the model and ship in Examples III. and IV., calculate in a similar manner (1) E.H.P., (2) B.H.P., (3) I.H.P. Also, the power expended on (a) engine friction, &c., (b) screw friction and slip, (c) skin (water) friction, (d) residuary resistance for a steamer 250 feet long, 31.2 feet beam, 15.6 feet draft, to go at 15 knots.

13. Experiments made by Froude to determine the skin resistance of planks in water give the following results:—

Speed in feet per minute = V ,	200	400	600	800
Total resistance per 100 sq. feet in lbs. = R ,	3.28	11.7	24.6	41.7

Test whether the relation between R and V can be expressed by a law of the type R varies as V^n , and, if so, find the values of f and n in the formula $R = f S V^n$, in which S = wetted surface in square feet. (C. & G., 1903, H., Sec. C.)

N.B.—See Appendices B and C for other questions and answers.

LECTURE VI.—A.M. INST. C.E. EXAM. QUESTIONS.

1. State briefly the laws of fluid friction deduced from the experiments of Froude. Taking skin friction to be 0.4 lb. per square foot at 10 feet per second, find the skin resistance in lbs. of a ship of 12,000 square feet immersed surface, at 15 knots (a knot = 6,086 feet per hour). Also the H.P. to overcome skin friction. (I.C.E., Oct., 1897.)

2. A clean-painted vessel, whose average waterline is 350 feet long and wetted surface 23,000 square feet, moves through the water at a speed of 12 knots; if the resistance in lbs. per square foot of surface at 6 knots be 0.227, and $\left(\frac{12}{6}\right)^{1.825} = 3.54$, find the resistance due to surface friction and the horse-power necessary to overcome that resistance. (I.C.E., Oct., 1897.)

3. Prove that if one vessel be a model of another n times as long, the indicated horse-power of the latter at a given speed will be $n^{\frac{5}{2}}$ times that of the former at the given speed divided by $n^{\frac{1}{2}}$, if the surface friction be assumed to follow the law of comparison, and the efficiency of engines and propeller be the same in both cases. (I.C.E., Oct. 1897.)

4. Give an approximate estimate of the I.H.P. necessary to drive a ship 300 feet long with a displacement of 2,400 tons (the block coefficient for displacement being 0.5) at a speed of 20 knots; and explain the process throughout. Show also what are the various methods that might be employed in practice for arriving at a trustworthy result, and their respective merits and defects. (I.C.E., Oct., 1897.)

5. Give a general account of the laws of fluid friction, stating especially what knowledge of them is due to the experiments of D'Arcy and Froude. (I.C.E., Feb., 1898.)

6. Assume any dimensions you please for a vessel, and show by numerical illustrations how you would make an approximate estimate of the I.H.P. necessary to drive her at a given speed upon whatever draught of water, displacement, and fineness of form you choose; and explain the nature and possible amount of error involved by the assumption upon which your estimate is based. (I.C.E., Feb., 1898.)

7. Explain the "law of comparison" for the resistance to motion in a fluid of similar bodies; and show how this can be applied in practice, being given a curve of progressive speed trials of a ship, to the determination of the I.H.P. requisite to drive another ship of different dimensions, which is similar in form, at a given speed. Discriminate between the causes of resistance that come under the law of comparison in such a case, and those which do not; and show how each are dealt with separately in your estimate of power. What other conditions of similarity, besides that of underwater form, require to exist in the two vessels, in order to reduce the causes of possible error to a minimum. (I.C.E., Feb., 1898.)

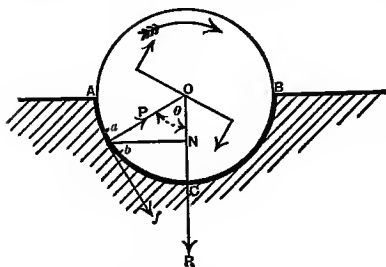
8. How does the coefficient of dry friction vary with the speed of rubbing? Does this explain why it is better not to skid the wheels when bringing a train to rest? (I.C.E., Oct., 1904.)

N.B.—See Appendices B and C for other questions and answers.

LECTURE VII.

CONTENTS.—Calculation of Work Lost by Friction in Journals—Example I.—Corrections for Twisting Moment on Crank Shaft of Engine—Rolling Friction—Tractive Force—Anti-Friction Wheels—Kynoch Roller Bearings—Ball Bearings—Ball Bearings for Cycles—Friction of Flat Pivots and Collar Bearings—Friction of Conical Pivots—Examples II. and III.—Schiele's Anti-Friction Pivot—Frictional Resistance between a Belt or Rope and a Flat Pulley—Example IV.—Resistance to Slipping of a Rope on a Grooved Pulley—Questions.

Calculation of Work Lost by Friction in Journals.—If Coulomb's laws of friction be applied to the case of cylindrical surfaces, then the frictional resistance to rotation of a cylindrical journal in its bearing (measured along its tangent to the surfaces in contact) would be $F = \mu R$; where, R is the total normal pressure acting on the journal. This would probably be the case if the journal and its bearing were in contact along a very narrow



FRICTION OF JOURNALS.

longitudinal area parallel to the axis of the journal. If, however, the journal and bearing are well worn and a good fit, then contact will take place over a considerable area, and the distribution of pressure will vary from point to point.

Suppose we consider a well fitting horizontal cylindrical journal and its bearing, acted on by vertical forces, the resultant of which is R .

Let p = intensity of the normal pressure on any longitudinal area—for example, $a b$.

„ f = the friction along the narrow longitudinal area at $a b$.

„ l = the length of the journal.

Then, $p \times a b \times l$ = Total normal pressure on the area $a b$.

Resolving vertically, we get:—

$$R = \sum (p \cos \theta \times a b \times l).$$

$$\text{Or,} \quad R = l \sum (p \cos \theta \times a b) \quad . \quad . \quad . \quad . \quad (1)$$

If, μ = Coefficient of friction between the journal and its bearing.

And, F = Total friction over the whole bearing.

Then, $F = \sum f = \mu l \sum p \times a b \dots \dots \dots (2)$

If we knew the law according to which p varies from point to point, it would be an easy matter to calculate the frictional resistance in this case. When the journal and its bearing are well worn and a good fit, we may assume that the intensity of pressure at any point will vary as the vertical distance of the point below the diameter AB :—

i.e., $p \propto ON$.

But this is the same law as that which would be followed by a heavy liquid enclosed in the semi-cylindrical space ACB , the total weight of the liquid being R .

We know, that in such a case, the total normal pressure on the cylindrical surface ACB , would be = *area of surface ACB \times depth of c.g. of surface below AB \times weight of a cubic unit of the liquid.*

Let, d = diameter of journal.

Then,
$$\left. \begin{array}{l} \text{Area of semi-cylindrical} \\ \text{surface } ACB \end{array} \right\} = A = \frac{1}{2} \pi d l.$$

And, $\text{Depth of c.g. of surface below } AB = \bar{x} = \frac{d}{\pi}.$

We now require to find the weight of a cubic unit of the hypothetical liquid in ACB .

Let, w = weight of a cubic unit.

Then,

$\text{Vol. of semi-cylinder } ACB \times w = R.$

$\therefore \frac{1}{2} \frac{\pi}{4} d^2 l \times w = R.$

Or,
$$w = \frac{8 R}{\pi d^2 l}.$$

Hence,
$$\left. \begin{array}{l} \text{The total normal pressure} \\ \text{over whole surface } ACB \end{array} \right\} = A \bar{x} w.$$

i.e.,
$$l \sum p \times a b = \frac{1}{2} \pi d l \times \frac{d}{\pi} \times \frac{8 R}{\pi d^2 l} = \frac{4}{\pi} R.$$

But from Equation (2) . . . $F = \mu l \Sigma p \times a b$.

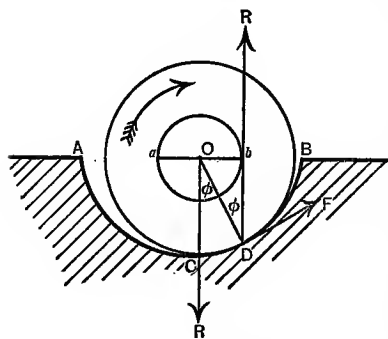
Hence, $F = \mu \frac{4}{\pi} R = \frac{4}{\pi} \mu R$.

∴ The frictional moment, $M = F \times \frac{d}{2} = \frac{2}{\pi} d \mu R$.

∴ Work lost in friction in one }
turn of journal } $= M \times 2 \pi = 4 d \mu R$. (I)

This result is only true on the assumption that the normal pressure varies as the depth of the point below A B, and since the surfaces are cylindrical, it is doubtful if μ has the same value as in the case of plane surfaces.

If the bearing be so well worn that its radius of curvature is slightly greater than that of the journal, we may suppose contact to occur along a narrow longitudinal area parallel to the axis of the journal. This state of affairs is shown in an exaggerated manner, by the annexed figure. The small area on which the journal bears, is not situated exactly at C (the lowest



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part of the bearing), but at or near D, such that the $\angle COD = \phi$, the "angle of friction." This is due to the shaft tending to climb out of its bearing. When the inclination of the surfaces in contact is equal to the angle of friction, then slipping takes place. This occurs when the shaft bears at D, since the tangent plane at that place is inclined at an angle, ϕ , to the horizon.

Let R = Resultant force acting on the shaft, which for the present may be supposed to act vertically downwards through the centre of the shaft. At D, introduce a force equal, parallel, and opposite in direction to R . Then this is the *resultant reaction* (i.e., the resultant of the *normal* reaction and friction) at the point D. Through O, draw the common perpendicular Ob , to the forces R . Then those two forces form a couple, whose moment = $R \times Ob$. This frictional moment resists the rotation of the shaft.

$$\therefore M = R \times O b.$$

$$\text{But,} \quad O b = O D \sin \varphi = \frac{d}{2} \sin \varphi.$$

$$\therefore M = \frac{1}{2} R d \sin \varphi.$$

Hence,

$$\left. \begin{array}{l} \text{Work lost in friction in} \\ \text{one turn of journal} \end{array} \right\} = M \times 2 \pi = R \pi d \sin \varphi. \quad (\text{II}_a)$$

$$\text{Where,} \quad \sin \varphi = \frac{\tan \varphi}{\sqrt{1 + \tan^2 \varphi}} = \frac{\mu}{\sqrt{1 + \mu^2}}.$$

For well lubricated journals φ is very small, so that we may assume that $\sin \varphi = \tan \varphi = \mu$.

$$\left. \begin{array}{l} \text{Then, Work lost in friction in} \\ \text{one turn of journal} \end{array} \right\} = M \times 2 \pi = \pi d \mu R. \quad (\text{II}_b)$$

For practical purposes it is more convenient to use formula (II_b) than (I) or (II_a), remembering that μ is a special coefficient for journals, to be determined by experiment.

We have seen, that the resultant reaction of the bearing at D is R and acts vertically upward. This reaction is tangential to a small circle, $a b$, which can be described about O, and is spoken of as the "*friction circle*." Let d_1 be the diameter of the friction circle.

$$\text{Then,} \quad R \times \frac{d_1}{2} = M = \frac{1}{2} d \mu R.$$

$$\therefore d_1 = \mu d \dots \dots \dots (\text{III})$$

i.e., Diameter of friction circle = $\mu \times$ diameter of journal.

Hence, in cases of journal friction it is often convenient to draw the friction circle, and then the direction of the resultant reaction of the bearing may be easily determined. This resultant reaction is always a tangent to the friction circle.

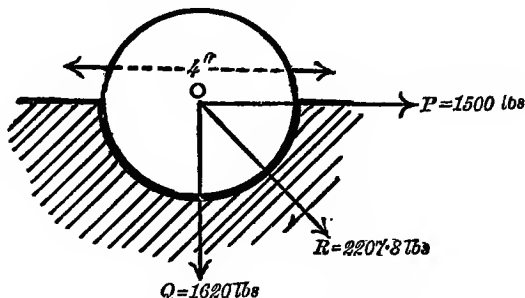
If the bearing has a cover, the resultant force, R, must be increased by the tension in the bolts holding down the cover when calculating M.

EXAMPLE I.—A horizontal shaft, 4 inches in diameter, resting in bearings at its ends, transmits power to various machines by means of belts passing over pulleys keyed to the shaft. The tension in the belts causes a horizontal force of 1,500 lbs., and a vertical downward force of 500 lbs. in a plane at right angles to the shaft. The weight of the shaft and pulleys is 10 cwts.

Coefficient of friction between the shaft and its bearings is 0.07. Find the horse-power lost in friction, the shaft making 100 revolutions per minute.

ANSWER.—The forces acting on the shaft are (1) a horizontal force of 1,500 lbs.; (2) a vertical force of $500 + 10 \times 112 = 1,620$ lbs. Since these forces act at right angles to each other, the resultant pressure on the bearings will be:—

$$R = \sqrt{P^2 + Q^2} = \sqrt{1500^2 + 1620^2} = 2207.8 \text{ lbs.}$$



FORCES ACTING ON A SHAFT.

Each bearing may or may not sustain equal shares of this resultant load. This will depend on the arrangement of the pulleys on the shaft. Nevertheless, we may consider only one bearing of the shaft, and suppose the resultant load on this bearing to be $R = 2207.8$ lbs. Hence, from the formula already obtained, we get:—

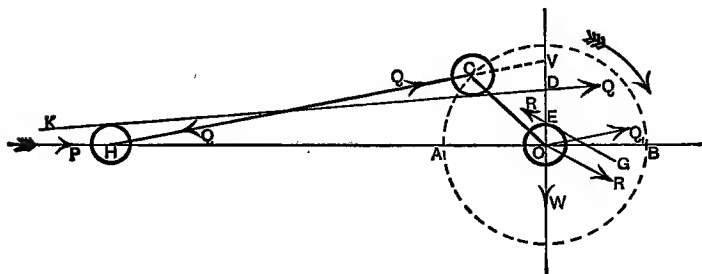
$$\left. \begin{array}{l} \text{Work lost in friction in} \\ \text{one turn of shaft} \end{array} \right\} = \pi d \mu R.$$

$$\begin{aligned} \therefore \text{H.P. lost in friction} &= \frac{\pi d n \mu R}{33,000}, \\ &= \frac{22}{7} \times \frac{4}{12} \times 100 \times .07 \times 2207.8, \\ &= .49. \end{aligned}$$

Corrections for Twisting Moment on Crank Shaft of Engine.—

Let OC be the crank, and HC the connecting-rod of an ordinary direct acting engine. Let P = total effective pressure on cross-head pin at H . Then, as has been shown in the author's *Text-*

Book on Steam Engines, the twisting moment on the crank shaft at O is $P \times OV$; where V is the point of intersection of the centre line of the connecting-rod with the line through O perpendicular to centre line, HB, of the engine. This method of calculating the twisting moment at any point of the stroke is adopted when we wish to neglect the friction of the journals at H, C, and O. We shall now determine the effect on the



FINDING THE TWISTING MOMENTS ON CRANK SHAFT.

twisting moment when the friction of the journals is taken into account.

Draw the friction circles for the journals at H, C, and O, as described above.*

Then the resultant pressure on a bearing must be tangential to its friction circle. Therefore, the thrust, Q , along the connecting-rod must be tangential to the friction circle at H , and also to the friction circle at C . If the student considers the direction of motion at H and C , he will observe, that the line of thrust (which is a common tangent to the circles at H and C) must be drawn as shown. Thus, the thrust is actually along KD instead of HV . This takes friction into account so far as the crosshead and crank-pin bearings are concerned.

Let W represent the total vertical load on the crank-shaft, including the weight of the shaft itself. The crank-shaft is then acted on by two forces, Q_1 and W , as shown, Q_1 being equal and parallel to Q along KD . The resultant of these two forces is R , and the reaction of the bearing, which is also R , acts in the direction, GR , parallel to OR , and tangential to the friction circle at O .

* On the figure the friction circles for these three journals have been drawn to a very much exaggerated scale for the sake of clearness.

Let G R cut O V at the point E. Then :—

Effective twisting moment on crank shaft = $P \times E D$.

For, from what has been already said:—

Twisting moment due to thrust Q along K D = $P \times O D$.

Where P is the horizontal component of Q, or, what is the same thing, P is the total pressure acting on the piston.

Similarly, since the horizontal component of R = horizontal component of Q = P.

∴ *The resisting moment due to R along G R is* = $P \times O E$.*

∴ *Effective twisting moment* $\left. \begin{array}{l} \text{on crank shaft} \end{array} \right\} = P(O D - O E) = P \times E D$. (IV)

It is evident, that as D approaches E, the effective twisting moment on the crank shaft diminishes, until when D coincides with E there will be no effective moment acting on the crank. This will occur at four positions of the crank; two on either side of centre line of the engine. The angle which the crank makes with the centre line of the engine when D coincides with E, is called the "*dead angle*," and when the crank lies within this angle (on either side of engine centre line), no pressure, however great, applied to the piston will move the engine. If the position of the crank had been taken in any other quadrant, say, in the quadrant V O B, then the direction of the thrust Q, would be such that K D, the common tangent to the friction circles at H and C, would be drawn *below* the centre line H C. The direction of the common tangent H D should, however, present no difficulty if the student only pays attention to the direction of rotation of the journal in its bearing.

Rolling Friction.—The resistance which is experienced when a wheel or cylinder is rolled along a rough horizontal plane is called *Rolling Resistance*, or *Rolling Friction*. This resistance is in general much less than sliding friction. It depends on the radius and breadth of the wheel and also on the nature of the surface over which the wheel rolls. It is found by experiment, that the resistance to rolling on a horizontal plane is expressed by a formula of the form

$$P = \frac{W}{r} c.$$

* The thrust R along E R is in a similar condition to the thrust Q along K D; that is to say, the friction circle at O can be looked upon as a small crank-pin circle and E R the direction of thrust of the connecting-rod.

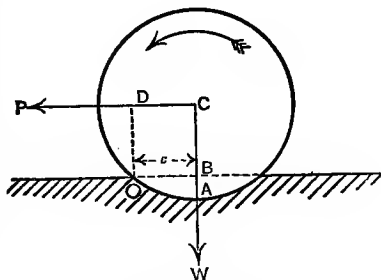
Where, P = Pull required to overcome the resistance (as measured by a horizontal force at the axis of the rolling body).

„ W = Total weight of the rolling body.

„ r = Radius of rolling body.

„ c = Constant *length* measured in the same units as r (depending on the nature of the surfaces in contact).

We can best explain the nature of this resistance, and how the above formula is obtained, by considering the case of a cylinder or broad wheel rolling along an ordinary road. In such a case, the wheel sinks into the ground and leaves a rut along its course. The depth of this rut will depend on the total weight, W , on the wheel, the radius, r , and the softness of the ground. The result of the sinking is, that the force, P , applied at C is employed in continually drawing the wheel over an obstacle at O , in front of the wheel.



ROLLING FRICTION.

Let the height of this obstacle (or, what is the same thing, the depth of the rut) be $BA = h$. Then, by taking moments about O , we get :—

$$P \times OD = W \times OB,$$

$$\therefore P = \frac{W}{OD} \times OB = \frac{W}{r - h} \times OB.$$

Put $OB = c$. Then, since h will, in general, be small compared with r , we may neglect it in the denominator on the right-hand side of the equation, and write

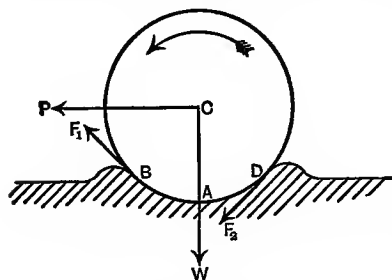
$$P = \frac{W}{r} c.$$

Hence, we see where the constant c comes from, and why it is a *length* of the same units as r .

From the results of the few experiments which have been carried out on rolling resistance on ordinary roads and rails, the

above formula appears to hold good, and c is found to be independent of r . Thus, when r is expressed in *inches*, c may be taken at from $\cdot 02$ inch to $\cdot 025$ inch for iron wheels and rails; $\cdot 06$ inch to $\cdot 1$ inch for iron wheels on wood, the lower values being taken for the harder woods. For carriage wheels on good macadamised roads c may be taken at $\cdot 5$ inch; but this value will vary considerably with the nature of the ground, being as high as 3 inches or even 5 inches with soft ground.

If the surface over which the wheel rolls be very soft and elastic the expression for the resistance is more complex and is very difficult of explanation. Take, for example, the case of a wheel rolling along a thick sheet of india-rubber. In such a



ROLLING FRICTION ON A
PLIABLE SURFACE.

case the rubber will take the form shown by the figure, being heaped up both in front of and behind the wheel. When the wheel moves it tends to surmount and compress the rubber in front at B. But the rubber at B tends to avoid this compression, and, as a consequence, heaps itself up in front as shown. During the action, the rubber "creeps"

over the surface of the wheel at B, and in doing so, a frictional resistance, F_1 , is set up which opposes the onward motion of the wheel. Again, as the wheel moves onward, the heaped-up rubber in the rear at D tends to regain its normal state of flatness of surface, and in doing so creeps down the surface of the wheel introducing a frictional resistance, F_2 , in the direction shown, which resistance also opposes the progress of the wheel. Thus the wheel is retarded by the two frictional resistances, F_1 and F_2 , in addition to the other force necessary to overcome the obstacle in front, as explained in the case of the wheel on ordinary ground.

Tractive Force.—By *tractive force* or *traction* is meant the effort necessary to draw a carriage or train along a level road or rail against the total frictional resistances. This total resistance includes axle and rolling frictions, the friction between the flanges of the wheels and the rails and the resistance of the air, &c.

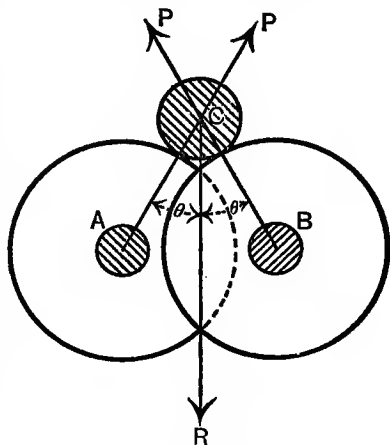
The following table gives the tractive force in lbs. per ton for different roads :—

TRACTION FORCE FOR DIFFERENT ROADS.

Paved roads,	33 lbs. per ton.
Macadamised roads,	44 to 67 " "
Gravel roads,	140 " 150 " "
Common earth roads,	200 " 250 " "
Railway trains (moving slowly),	2 " 8 " "

In the case of railway trains, the resistance of the air increases very rapidly with the speed of the train and with the velocity of the wind, so that at high speed the tractive force required to keep the train moving along a horizontal line may be as high as 50 lbs. per ton.

Anti-Friction Wheels. — These wheels are so arranged that their circumferences form a bearing for an axle or shaft. The axles A and B of the two anti-friction wheels are placed near to each other, and the axle, C, whose friction it is desired to reduce, rests on their circumferences. By this means, the resistance to the rotation of C is greatly diminished, and hence the arrangement is often resorted to in the case of philosophical and other delicate apparatus, where the frictional resistances have to be reduced to a minimum. The contrivance is neither sufficiently simple nor compact and strong to be adopted for the ordinary bearings of large heavy shafts; but a modification thereof is often met with in the ball bearings of the best constructed driving shafts for foot lathes, for American electric elevators, and for bicycles, tricycles, &c.* With ball bearings the friction is wholly that of rolling friction, but in the ordinary anti-friction wheels the rolling friction may



ANTI-FRICTION WHEELS.

* See Lecture X., page 109, of the author's *Elementary Manual of Applied Mechanics* for these applications in the cases of Sir Wm. Thomson's Siphon Recorder and Atwood's Machine, &c. Also the *Electrical Engineer* of New York, Nov. 2nd, 1892, for a description of the ball bearings in the Sprague-Pratt Electric Elevator for High Service Duty.

be neglected and then the friction of the axles, A and B, considered in the following way :—

Let R = Resultant vertical force acting on axle C.

„ D = Diameter of each wheel A and B.

„ δ = „ „ axle A and B.

„ d = „ of the shaft C.

„ 2θ = Angle A C B.

„ P = Pressure on A or B due to R on C.

„ μ = Coefficient of friction between axles A and B and their bearings.

Then, since R must be the resultant of P at A and P at B, we get:—

$$R = 2 P \cos \theta.$$

$$\text{Or, } P = \frac{R}{2 \cos \theta}.$$

Suppose the wheels A and B to make one complete turn. Then C will make n turns, so that:—

$$\pi d n = \pi D.$$

$$\text{Or, } n = \frac{D}{d}.$$

Hence, neglecting friction between C and circumferences of wheels A and B, we get:—

$$\left. \begin{array}{l} \text{Work absorbed in friction} \\ \text{in one turn of wheels A} \\ \text{and B} \end{array} \right\} = \left\{ \begin{array}{l} \text{Work absorbed in friction at axle} \\ \text{A + Work absorbed in friction} \\ \text{at axle B.} \end{array} \right.$$

$$= 2 \pi \delta \mu P.$$

$$= \pi \delta \frac{\mu R}{\cos \theta}.$$

But, had the axle C been resting in an ordinary bearing instead of on the circumferences of the wheels A and B, the

$$\text{Work absorbed in friction would be} = n \pi d \mu R,$$

$$= \pi D \mu R.$$

$$\therefore \frac{\text{Friction with anti-friction wheels}}{\text{Friction without anti-friction wheels}} = \frac{\pi \delta \frac{\mu R}{\cos \theta}}{\pi D \mu R} = \frac{\delta}{D \cos \theta}.$$

In general, $\theta (\frac{1}{2} \angle A C B)$ will be small, so that $\cos \theta$ will be unity very nearly.

$$\therefore \frac{\text{Friction with anti-friction wheels}}{\text{Friction without anti-friction wheels}} = \frac{\text{dia. of axle A or B}}{\text{dia. of wheel A or B}} = \frac{\delta}{D}.$$

This result agrees closely with experiments carried out on the friction of anti-friction wheels.

Kynoch Roller Bearings.—The rollers are made from a special quality of sheet steel, and are formed by rolling a blank of the shape shown by Fig. 1 into a hollow cylinder as seen at Fig. 2.

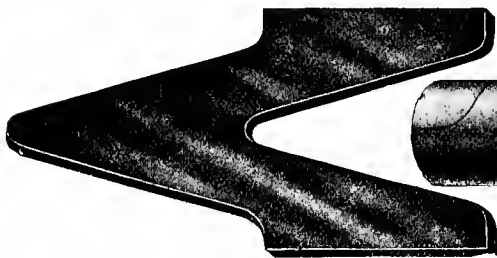


FIG. 1.—SHEET STEEL BLANK.



FIG. 2.—HOLLOW CYLINDER ROLLER.

The advantage of this construction for a roller, being, that the fibre or grain of the steel is always in the direction of rotation, which gives them an amount of elasticity that solid soft rollers do not possess. The rollers are not “hardened,” and therefore do not laminate the shaft.



FIG. 3.—CAGE OR GRID FOR KEEPING THE CYLINDRICAL ROLLERS PARALLEL TO THE AXIS OF THE SHAFT.

The rollers should always be kept parallel to the axis of the shaft. This is done by enclosing them in a guiding cage like Fig. 3, which is divided into three equal spaces. It can revolve

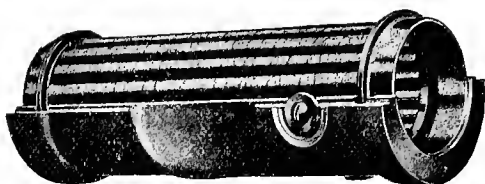


FIG. 4.—ROLLER BEARING WITH TOP CAP REMOVED.



FIG. 5.—ROLLER BEARING INSIDE ITS CASING.

easily on the shaft and within its casing, as seen by Figs. 4 and 5. The complete plummer block and its roller bearing is illustrated by Fig. 6.

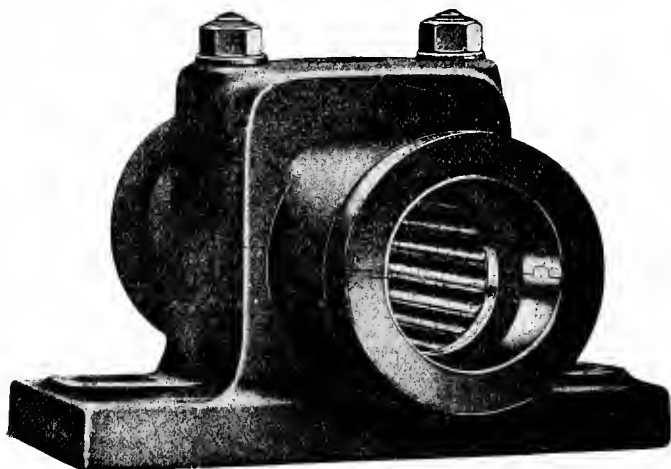


FIG. 6.—COMPLETE KYNOCH ROLLER BEARING AND PLUMMER BLOCK.

From actual tests, it has been found, that the coefficient of friction between a shaft and these roller bearings is only $\cdot 0115$

for repose, and $\cdot 0082$ at the normal speed of rotation. This is claimed-to be about one-eighth of the coefficient ($\cdot 07$) for the same shaft when run in plain bush bearings with ordinary lubrication. (*See Lecture VI., p. 78, at the top.*)

The makers of these bearings claim a saving in labour, oil, and in the transmission of power by their use.

Ball Bearings.—The student should again refer to Lecture VI. (p. 84, &c.) and revise the conclusions derived from the experiments on the “Friction of Journals,” as well as what has been said in this Lecture upon “Rolling Friction” and “Anti-friction Wheels,” before considering the present cases of ball and roller bearings.

It has been explained, that the work lost in friction *must*, in all cases, be proportional to the distance through which the friction acts. Hence, if by any convenient means we can reduce this distance (without at the same time increasing the other quantities which make up the total losses due to friction), and introduce the distinctly anti-friction qualities of *rolling* resistance, instead of the more wasteful *sliding* resistance, we shall save power in driving machines as well as wear and tear in their bearings.

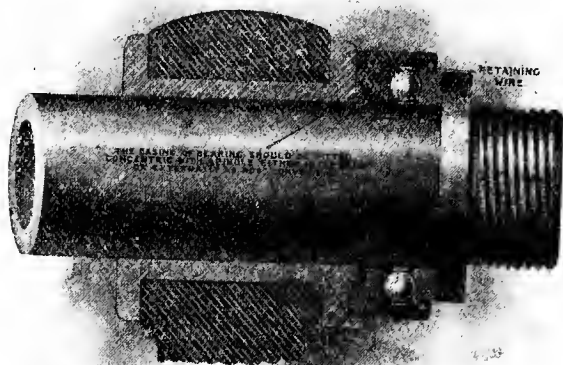
For instance, the wheels of a dog-cart are made large in diameter, that a horse may the more easily pull the vehicle over stones and the other obstacles to be met with on ordinary roads; but the axle bearings are made small with the object of minimising the distance through which their friction acts, for a given length traversed by the vehicle. And, as we shall presently see, axle friction can be still further reduced by the use of ball bearings, as has been done in the cases of bicycles, tricycles, and motor cars.

With ball bearings, rolling friction takes the place of the sliding friction in ordinary bearings. Consequently, the chief point to be attained in designing ball bearings is to get true rolling motion of the balls without any grinding action.

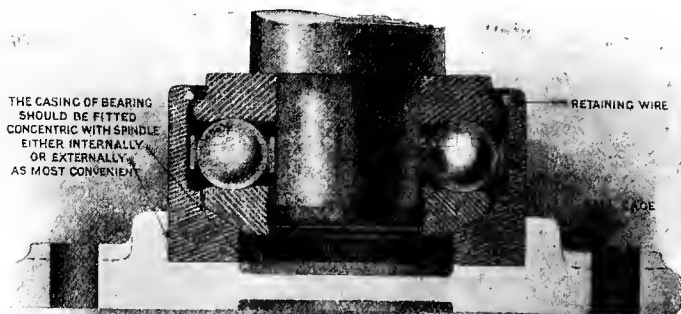
As a good example of various practical applications to machines, we have selected for illustration those by the Hoffmann Manufacturing Co., Chelmsford, who have made a special study of this subject. This firm have subdivided their bearings into what they term *Light*, *Medium*, and *Heavy* types.

The *Light Type* is designed for such cases where end thrust is only a part of the load on the shaft—*e.g.*, in the spindles of drilling machines, feed and elevating screws, and headstocks of lathes, as shown by the accompanying figure. The several parts of the bearing consist of—(1) balls, (2) case or housing, (3) hardened steel floating disc, (4) hardened steel cup, (5) ball

cage, (6) retaining wire, which latter keeps the other parts together. The floating disc is made of such a shape that it is automatically self-adjusting within the outer case. This quality,



BALL BEARING FOR A LATHE HEADSTOCK.



FOOT-STEP BEARING.

if properly carried out, ensures an equal load upon all the balls. It is recommended, that all such ball bearings should be lubricated with a heavy lubricant or motor grease. The grease acts like a lacquer, and ensures that an oily film is present between the various points of rolling contact. Oil should not be used, more especially for high speeds, as it naturally washes out the grease with which the bearings have been filled.

The *Medium Type* is more suitable for positions where neither the end thrust nor the speed are very great, such as for the footstep bearings of certain vertical shafts. In the previous figure the ball cage is clearly shown, and is generally used with all kinds of ball bearings, as it keeps the balls running in a definite path.

The *Heavy Type* is most applicable where the end thrust is a maximum for the size of shaft; such as the end thrust of worm gears, centrifugal machines, pivots, and crane hooks, as shown by the following figure. The ball races are formed by a ground



BALL BEARING FOR CRANE HOOK.

concave groove upon the working faces of the hardened steel cup and the floating disc or ring. These, as well as the balls, are made from special qualities of steel, and are hardened in such a way that, although their working faces are glass hard, their interior is toughened to withstand the crushing forces and shocks to which they may be subjected.

TABLE OF DIMENSIONS AND SAFE WORKING LOADS, &c.
With the Hoffmann Ball Bearings.

Type of Bearing.	Dia. of Shaft in Inches.	Revolutions per Minute.	Dia. of Balls in Inches.	Number of Balls.	Safe Working Load in Lbs.	Maximum Load for Crane Pivots or Hooks.
Light.	1	50 to 1,000	$\frac{1}{4}$	12	250 to 66	1,100 lbs.
	to 18	„ „	to 3	to 18	100,000 to 27,000	200 tons.
Medium.	$\frac{1}{2}$	50 to 1,500	$\frac{9}{32}$	8	610 to 120	1,700 lbs.
	to 9	„ „	to 3	to 10	100,000 to 20,000	130 tons.
Heavy.	$\frac{1}{2}$	50 to 2,000	$\frac{7}{32}$	6	130 to 20	280 lbs.
	to 6	„ „	to 3	to 8	100,000 to 16,000	100 tons.

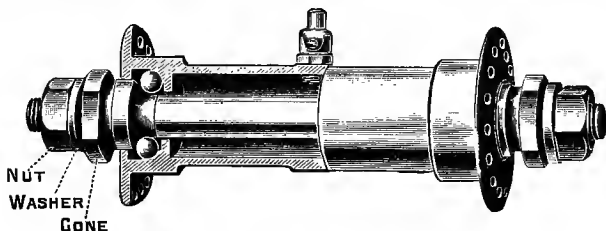
As may be seen from the above Table, which is a short extract from the more detailed one issued by the makers of these bearings, very great ranges of speed and safe working loads are guaranteed. When we consider the numerous cases wherein a minimum of starting effort, of tractive effort, and of torque are desirable, combined with freedom from noise, and economy in lubrication, it will be seen how wide the application of really well designed, proportioned, and satisfactorily made ball bearings has become.

Ball Bearings for Cycles.—The accompanying figures serve to illustrate the application of ball bearings to several of the more important parts in first-class cycles.*

In order to adjust the ball bearings in this hub, the left-hand hexagon nut *only* of the spindle, is first slackened back a little and the cone spanner applied to the thin nut part of the left-hand cone, to tighten or slacken the latter as required. Correct

* The second and fifth figures were specially drawn by the makers, Lea & Francis, Ltd., Coventry, from which electros were made by my Publishers for this book. The first, third, and fourth figures are from electros by Humber, Ltd., Coventry, whilst the last set of figures is from an electro by the first-named makers.

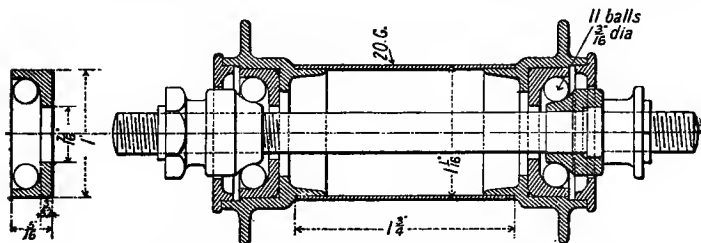
adjustment may be tested for by raising the wheel from the ground and by grasping it at the rim, trying to move it laterally. The wheel should be given a quarter of a turn, and tried again, and so on. Then spin the wheel to see if it runs freely, and finally settles with the oil cup at its lowest position. If the tyre is fitted, the wheel should settle with the valve at its



FRONT HUB OF A CYCLE.
(Showing Ball Bearing and Central Oil Cup.)

lowest position, or nearly so, irrespective of the position of the oil cup.

In the following complete longitudinal section, it will be seen that the balls rotate between their races on the spindle cones



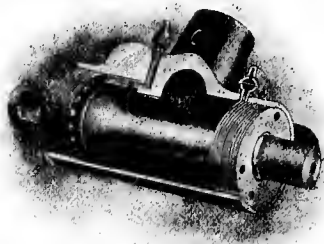
COMPLETE SECTION THROUGH FRONT HUB.
(Showing Balls, Ball Races, Cones, and Cone Adjustment.)

and those on their cups, in accordance with "two point contact plan." A section of a cup is shown separately by the left-hand figure. These cups are carefully turned from mild steel to fit the bored flanges, and after being hardened they are driven into their places.

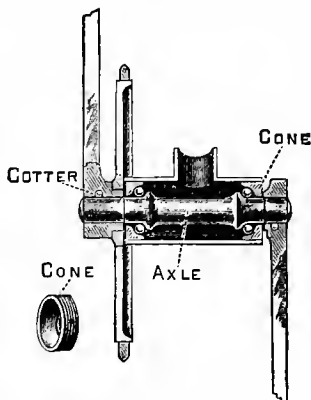
The third figure is specially given here to show one system of lubricating the crank bracket ball bearings. Small lubricators are attached to tubes which convey oil directly to the bearings. By removing a small screw from underneath the

bracket, the spent or dirty oil may be removed and the whole bearing flushed through with benzene, so as to thoroughly clean it.

The fourth figure serves to illustrate how the crank shaft ball bearings may be adjusted by screwing forward or backward the right- or the left-hand screwed cone (also shown separately). By unscrewing a locknut and withdrawing the tapered pin or



ORDINARY CRANK BRACKET HUB.
(Showing Method of Lubricating Ball Bearings.)



ORDINARY CRANK HUB.
(Showing Cone, Ball Bearing, and how to remove Cranks.)

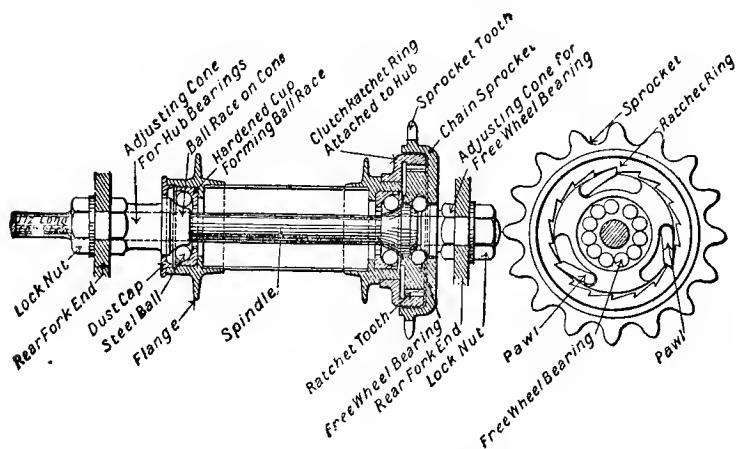
cotter from the left-hand crank, the latter may be removed together with the sprocket wheel. Thereafter the left-hand cone may be unscrewed and the balls on that side examined. In the same way those on the right-hand side may be directly inspected.

The fifth figure has been specially drawn to show every detail of an improved clutch and free wheel arrangement. Here the chain sprocket wheel ball race bearing is of unusually small diameter, in order to lessen the required number of balls and consequently the friction of this part. This clutch bearing is of the "four point contact pattern." When "free-wheeling," the bearing of this clutch being mounted on the non-rotating hub-axle, does not come into action. The only parts in contact are the small pawls between the stationary chain ring and the ratchet ring attached to the hub.

The left-hand view of the last set of figures illustrates the chain ring with its ball bearing. The right-hand figure shows an end view of the hub with the ratchet ring screwed into its

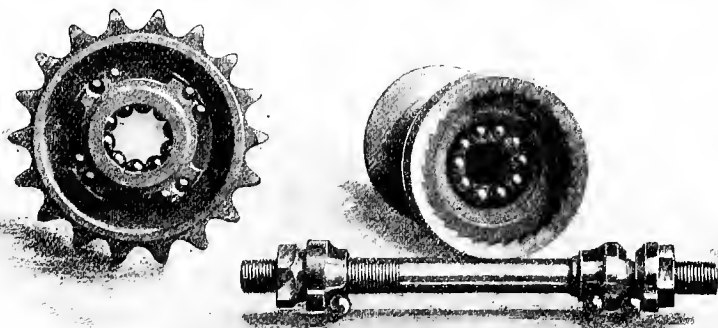
position. The axle serves to indicate the relative positions of the three rows of balls and their respective cone races.

Prof. Goodman states in his book on Mechanics, "he has found by experiment that the friction of ball bearings held an intermediate position between the friction of a plain bearing with syphon lubrication and that with bath lubrication. But,



REAR FORKED END HUB.

(Showing Free Wheel with Lea-Francis Clutch, having a Ball Race of Small Diameter.)



SEPARATE DETAILS OF THE LEA-FRANCIS CLUTCH.

(Showing Chain Ring, End View of Hub with Ratchet Ring, and the Axle with the three Ball Bearings.)

If the bearing be an annular ring or collar of outside radius, r_1 , and inside radius, r_2 , then:—

$$p = \frac{R}{\pi(r_1^2 - r_2^2)}.$$

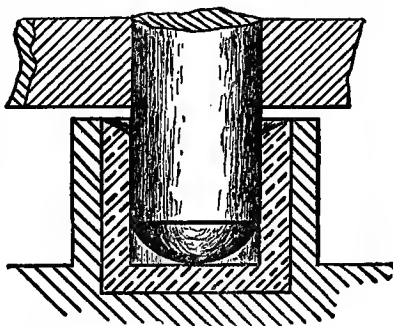
And
$$M = 2 \pi \mu p \int_{r_2}^{r_1} x^2 dx.$$

Or,
$$M = 2 \frac{\mu R}{r_1^2 - r_2^2} \times \frac{r_1^3 - r_2^3}{3}.$$

i.e.,
$$M = \frac{2}{3} \left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2} \right) \mu R \left\{ \begin{array}{l} \text{. (V)} \\ \text{Or, } M = \frac{1}{3} \left(\frac{d_1^3 - d_2^3}{d_1^2 - d_2^2} \right) \mu R \end{array} \right.$$

Where d_1, d_2 refer to the inside and outside diameter respectively.

Hence, Work lost in friction $\left\{ \begin{array}{l} \text{in one turn of} \\ \text{collar journal} \end{array} \right\} = M \times 2 \pi = \frac{2}{3} \pi \frac{d_1^3 - d_2^3}{d_1^2 - d_2^2} \mu R. \text{ (VI)}$



ROUNDED PIVOT.

In the case of a flat pivot $d_2 = 0$, and then:—

$$M = \frac{1}{3} d_1 \mu R. \text{ (Va)}$$

\therefore Work lost in friction in $\left\{ \begin{array}{l} \text{one turn of flat pivot} \end{array} \right\} = \frac{2}{3} \pi d_1 \mu R. \text{ (VIa)}$

Thus, we see that the frictional moment of a flat pivot in its footstep is only $\frac{1}{2}$ of that for a horizontal journal of equal diameter.

By diminishing the diameter of the pivot, the frictional moment will be diminished in the same proportion. Hence, we often find that small pivots are rounded at their lower ends and rest on a flat step in the manner shown by the accompanying figure.

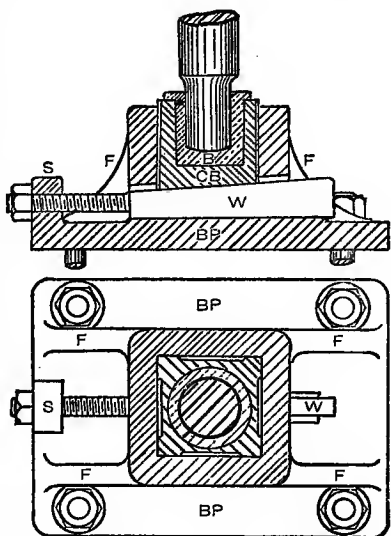
In the case of turn-tables and the vertical posts of cranes where the motion is slow this plan is also adopted. But for the bearings and footsteps of large shafts carrying a heavy load and moving at high speeds, the following style of adjustable footstep and bearing has been found best.

Friction of Conical Pivots.

— Sometimes the end of the vertical shaft is made conical—instead of flat or hemispherical—and turns in a step or bearing of corresponding shape. In this case the expression for the frictional moment will be different from the one just obtained for a flat pivot.

Let A D E B represent a section of a conical pivot, or frustum of a cone, with r and r_1 for its greatest and least radii respectively.

Let the total axial thrust, R , be uniformly distributed over the transverse section at A B. The manner in which this pressure is distributed over the step is not definitely known. The normal pressure may be constant at all points on the step, or it may vary according to some other law. If, however, we assume that the step always remains a good fit for the pivot,



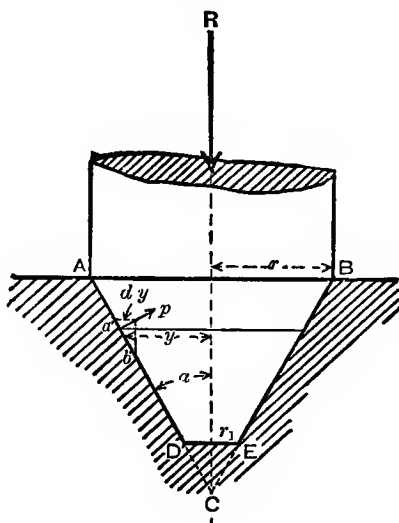
ADJUSTABLE FOOTSTEP.

INDEX TO PARTS.

B	represents	Bearing.
C B	„	Cast bush.
F	„	Footstep.
B P	„	Base plate.
W	„	Wedge.
S	„	Screw.

and that the wear is uniform throughout, we can then explain how the pressure, p , varies from point to point.

The wear over any small conical surface on the step depends on the product of the normal pressure, p , and the velocity of



CONICAL PIVOT.

rubbing, v . Hence, if the wear be everywhere the same, we must have:—

$$pv = \text{a constant.}$$

But the velocity of rubbing depends on the distance of the small area in question from the axis of rotation. If this distance be y , then:—

$$pv \propto py.$$

∴

$$py = \text{a constant.}$$

Consequently, the pressure at any point on the step varies inversely as the distance of the point from the axis of rotation.

Hence, if we consider a small conical area of breadth or slant length, \overline{ab} , and mean radius, y , and if the projection of \overline{ab} on a

horizontal plane be denoted by dy ; then, if α = half the conical angle ACB ; $dy = \bar{a}\bar{b} \sin \alpha$.

$$\left. \begin{array}{l} \text{Total normal pressure} \\ \text{on elementary frustum} \\ \text{area bounded by } a b \text{ as} \\ \text{a slant side} \end{array} \right\} = p \times 2\pi y \times \bar{a}\bar{b} = 2\pi p y \times \bar{a}\bar{b}.$$

Now, the sum of the vertical components of all such normal pressures must balance R .

$$\therefore R = \sum 2\pi p y \bar{a}\bar{b} \sin \alpha = 2\pi p y \sin \alpha \sum \bar{a}\bar{b}.$$

Since py has previously been shown to be a constant it may be written outside the summation sign.

But, $\sum \bar{a}\bar{b}$ is clearly = AD .

$$\text{And, } AD = \frac{r - r_1}{\sin \alpha}.$$

$$\therefore \sum \bar{a}\bar{b} = \frac{r - r_1}{\sin \alpha}.$$

$$\text{Consequently, } R = 2\pi p y (r - r_1).$$

$$\text{And, } \therefore py = \frac{R}{2\pi(r - r_1)}.$$

Substituting this value of py in equation (1), we get:—

$$\left. \begin{array}{l} \text{Total normal pressure} \\ \text{on elementary frustum} \\ \text{area} \end{array} \right\} = \frac{R}{r - r_1} \bar{a}\bar{b} = \frac{R}{(r - r_1) \sin \alpha} dy.$$

$$\therefore \left. \begin{array}{l} \text{Moment of friction for} \\ \text{elementary frustum} \\ \text{area} \end{array} \right\} = \frac{\mu R}{(r - r_1) \sin \alpha} y dy.$$

$$\text{And, } \left. \begin{array}{l} \text{Moment of friction} \\ \text{for whole bearing} \end{array} \right\} = \frac{\mu R}{(r - r_1) \sin \alpha} \int_{r_1}^r y dy.$$

$$\text{Or, } M = \left(\frac{\mu R}{(r - r_1) \sin \alpha} \right) \frac{r^2 - r_1^2}{2}$$

$$\text{i.e., } M = \frac{1}{2} \frac{r + r_1}{\sin \alpha} \mu R. \quad \dots \quad (\text{VII})$$

$$\therefore \left. \begin{array}{l} \text{Work lost in friction in} \\ \text{one turn of conical} \\ \text{pivot} \end{array} \right\} = M \times 2\pi = \frac{\pi(r + r_1)}{\sin \alpha} \mu R. \quad (\text{VIII})$$

If the pivot comes to a point at O, then $r_1 = 0$, and then:—

$$M = \frac{1}{2} \frac{r}{\sin \alpha} \mu R. \quad \dots \quad (IX)$$

EXAMPLE II.—A vertical shaft, 4 inches in diameter, turns on a flat pivot. The weight of the shaft with its wheels, &c., is 2,500 lbs. Find the horse-power lost in friction between the end of the shaft and its footstep, the shaft making 140 revolutions per minute, and the coefficient of friction being taken at 0.08.

ANSWER.—Here $d = 4'' = \frac{1}{3}$ ft., $n = 140$, $R = 2,500$ lbs.

Hence, assuming formula VI_a, we get:—

$$\left. \begin{array}{l} \text{Work lost in friction} \\ \text{in one turn of} \\ \text{pivot} \end{array} \right\} = \frac{2}{3} \pi d \mu R.$$

$$\therefore \text{H.P. lost in friction} = \frac{\frac{2}{3} \pi d n \mu R}{33,000}$$

$$\begin{aligned} \text{"} \quad \text{"} \quad &= \frac{\frac{2}{3} \times \frac{22}{7} \times \frac{1}{3} \times 140 \times .08 \times 2,500}{33,000} = .59. \end{aligned}$$

EXAMPLE III.—If, in the last example, the pivot had been conical instead of flat, the angle at the vertex of the cone being 60° , and the smallest diameter of the pivot $1\frac{1}{2}$ inches, what would then be the H.P. lost in friction, assuming the pressure on the step to vary according to the law stated in the text?

ANSWER.—Here $d_1 = 1\frac{1}{2}''$, $\alpha = 30^\circ$, the other data being same as in last example. Then, by formula VIII. :—

$$\left. \begin{array}{l} \text{Work lost in friction} \\ \text{in one turn of pivot} \end{array} \right\} = \frac{\pi (r + r_1)}{\sin \alpha} \cdot \mu R.$$

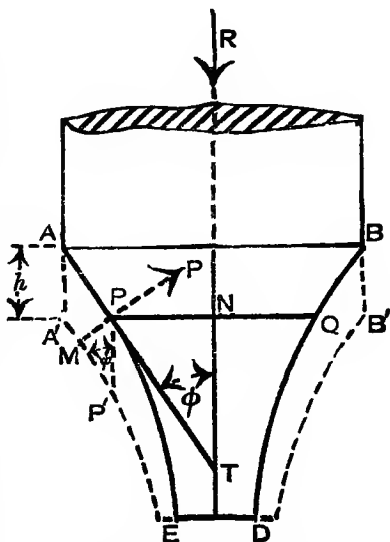
$$\text{"} \quad \text{"} \quad = \frac{\pi (d + d_1)}{2 \sin \alpha} \cdot \mu R.$$

$$\therefore \text{H.P. lost in friction} = \frac{\pi (d + d_1) n}{2 \sin \alpha} \cdot \frac{\mu R}{33,000}$$

$$\begin{aligned} \text{"} \quad \text{"} \quad &= \frac{\frac{22}{7} \times \left(\frac{4}{12} + \frac{1\frac{1}{2}}{12} \right) \times 140 \times .08 \times 2500}{2 \times \frac{1}{2} \times 33,000} \\ \text{"} \quad \text{"} \quad &= 1.2. \end{aligned}$$

Schiele's Anti-Friction Pivot.—The assumption made in our last investigation regarding the uniform wear of the ordinary conical pivot and step is not strictly correct. When such pivots have been at work for some time it is found that the contact between the pivot and its step is very imperfect, due to the unequal wear which naturally arises from the difference of velocities at different parts of the conical surface. Since this is a matter of some importance, especially with certain kinds of instruments and machines, we shall here investigate the proper form to be given to the pivot and its step in order that the vertical wear of the latter may be everywhere the same.

The figure represents a section of such a pivot. During wear of the step let the pivot sink through a vertical depth, $A A' = h$. Then, by hypothesis, the vertical wear everywhere will be h ; so that any point, P , will, after wear, be at P' , where $PP' = h$. The dotted curves represent the outline of the pivot or step after wear, and it is evident that they will be similar to the full curves $A P E$, $B Q D$.



SCHIELE'S ANTI-FRICTION PIVOT.

Consider a point, P , on the curve. Draw PN perpendicular to the axis of the shaft. Let PT be the tangent to the curve at the point P , and PM the normal at that point.

Let y = Ordinate PN .

„ ω = Angular velocity of shaft.

„ p = Intensity of normal pressure.

„ μ = Coefficient of friction between pivot and step.

The normal wear at P per unit area is:—

$$PM \propto \mu p \omega y.$$

Now, for a small vertical wear, we may consider the triangles TPN , $P'PM$, as similar.

$$\therefore PT : PN = PP' : PM.$$

$$\text{Or, } PT = \frac{PP' \times PN}{PM},$$

$$" = \frac{h \times y}{\mu p \omega y}.$$

In this pivot, the vertical wear is supposed to be everywhere the same, therefore, the intensity of normal pressure, p , must be constant for all points on the pivot.

$$\therefore PT = \frac{h}{\mu p \omega} = \text{a constant.}$$

Consequently, the curve is such, that the length of the tangent, PT , at any point, P , is constant. The curve having this property is known as the "*Tractrix*" or "*Tractory Curve*." Hence, a suitable form of pivot is obtained by the revolution of such a curve about its axis. This form of pivot was invented by C. Schiele, and is called "*Schiele's Anti-Friction Pivot*." Such pivots are well adapted for high-speed machinery, the wear being perfectly uniform throughout and giving a very smooth motion.

Calculation of Friction Moment in Schiele's Pivot.

Let R = Total axial thrust on shaft.

" r = Largest radius of pivot = radius of shaft.

" r_1 = Smallest " "

$$\text{We shall now show, that } p = \frac{R}{\pi(r^2 - r_1^2)}.$$

Consider a small conical area of the pivot, the mean diameter of which is $PQ = 2y$, and length of slant side, ds .

$$\text{Then, } \text{Area of elementary ring} = 2\pi y ds.$$

$$\text{But, } ds = \frac{dy}{\sin \phi}.$$

$$\therefore \text{Area of elementary ring} = \frac{2\pi}{\sin \phi} y dy.$$

$$\text{Or, } \left. \begin{array}{l} \text{Total normal pressure} \\ \text{on elementary ring} \end{array} \right\} = P = \frac{2\pi p}{\sin \phi} y dy.$$

Resolving vertically, we get :—

$$\text{Vertical component of } P = P \sin \phi = 2 \pi p y dy.$$

$$\therefore R = 2 \pi p \int_{r_1}^r y dy = \pi p (r^2 - r_1^2).$$

$$\text{And, therefore, } p = \frac{R}{\pi (r^2 - r_1^2)}.$$

Next, the moment of friction for the elementary ring is :—

$$dM = \mu P y = \frac{2 \pi \mu p}{\sin \phi} y^2 dy.$$

We have seen that the length of the tangent, PT , at any point, P , on the curve is constant, hence :—

Let t = Length of tangent, PT .

$$\text{Then, } t = \frac{y}{\sin \phi}$$

$$\therefore dM = 2 \pi t \mu p y dy.$$

$$\text{Or, } M = 2 \pi t \mu p \int_{r_1}^r y dy.$$

$$\therefore M = 2 \pi t \times \frac{\mu R}{\pi (r^2 - r_1^2)} \times \frac{r^2 - r_1^2}{2}.$$

$$\text{Or, } M = \mu R t \quad \dots \dots \dots (X)$$

an equation which shows that the friction-moment depends only on the thrust along the shaft and the length of the constant tangent.

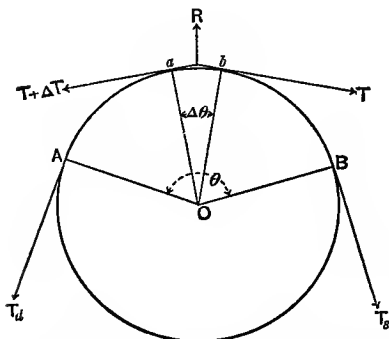
The minimum length of the tangent is $t = r$, for then the line, AB , is a common tangent to the curves, APE , BQD . Hence :—

$$M_{min} = \mu R r \dots \dots \dots (X_a)$$

The friction-moment with this pivot is thus one-half greater than with a flat pivot of the same diameter, but, as already said, the wear is more uniform throughout the bearing surface.

Frictional Resistance between a Belt or Rope and a Flat Pulley.

—Let the figure represent a belt stretched over a pulley. Let T_d , T_s denote the tensions in the two parts of the belt not in contact with the pulley. Suppose the belt to be just on the point of slipping on the pulley in the direction B to A, so that



$$T_d > T_s.$$

Let $\theta = \text{A O B} = \text{angle subtended at centre, O, by part of belt in contact with pulley}$; and let $\mu = \text{co-efficient of friction between belt and pulley rim}$.

FRICTION BETWEEN BELT AND PULLEY.

Consider the equilibrium of the part of the belt on a very small arc, ab , anywhere between A and B, the arc, ab , subtending an angle, $\Delta \theta$, at the centre, O, of the pulley.

Let the tensions in the belt at a and b be $T + \Delta T$, and T respectively, so that the increase in tension over the arc, ab , is ΔT . The directions of $T + \Delta T$ and T will be along the tangents at a and b respectively.

Let $R = \text{Resultant reaction between part of belt, } ab, \text{ and the pulley rim, due to tensions in belt at } a \text{ and } b$.

Then, neglecting the small difference, ΔT , between these tensions, we get:—

$$R = 2 T \cos \left(90^\circ - \frac{\Delta \theta}{2} \right) = 2 T \sin \frac{\Delta \theta}{2}.$$

Since $\Delta \theta$ is a very small angle, we may write:—

$$\sin \frac{\Delta \theta}{2} = \frac{\Delta \theta}{2} \text{ very approximately.}$$

$$\therefore R = T \Delta \theta.$$

But, since slipping is about to take place, ΔT must be a measure of the friction over the arc, ab .

$$\therefore \Delta T = \mu R = \mu T \Delta \theta.$$

$$\text{Or, } \frac{\Delta T}{T} = \mu \Delta \theta.$$

Hence, in the limit, when the arc, ab , is indefinitely small, we get:—

$$\frac{dT}{T} = \mu d\theta.$$

Therefore, for the whole arc, AB , of contact, we get:—

$$\int_{T_s}^{T_d} \frac{dT}{T} = \mu \int_0^\theta d\theta.$$

$$\text{i.e.,} \quad \log_e T_d - \log_e T_s = \mu \theta.$$

$$\left. \begin{array}{l} \text{Or,} \\ \text{Or,} \end{array} \quad \begin{array}{l} \log_e \frac{T_d}{T_s} = \mu \theta \\ \frac{T_d}{T_s} = e^{\mu \theta} \end{array} \right\} \dots \dots \dots \text{(XI)}$$

Since $e = 2.7182$ (the base of the Napierian system of logarithms) it is a constant; and since μ is also a constant, we may write the last equation in the form

$$\frac{T_d}{T_s} = k^\theta. \dots \dots \dots \text{(XII)}$$

thus showing that:—The ratio of the tensions increases as the power of the number representing in circular measure the angle subtended by the belt at the centre of the pulley.

The ratio is thus independent of the radius of the pulley.

The above results are true for a rope wound round a post and also for friction brakes wherein the strap encircles the friction wheel or pulley.

From these results we can understand why a sailor has such a power of holding back a ship at a quay by merely coiling the rope two or three times round a post. For example, when a rope is coiled *once* round a post, let $T_d/T_s = 4$. Then, when coiled *twice* round the post, $T_d/T_s = 4^2 = 16$; when *thrice* round the post, $T_d/T_s = 4^3 = 64$, and so on. This shows how rapidly the resistance increases.

In the formula $\log_e \frac{T_d}{T_s} = \mu \theta$ we may change into common logarithms by multiplying both sides by 0.434.

$$\therefore \quad \text{Log} \frac{T_d}{T_s} = 0.434 \mu \theta. \dots \dots \dots \text{(XIII)}$$

* θ in these equations is always expressed in circular measure.

EXAMPLE IV.—A rope is wound thrice round a post, and one end is pulled with a force of 20 lbs., what is the greatest pull which can be resisted on the other end of the rope when the coefficient of friction is 0.4?

ANSWER.—Here $T_s = 20$ lbs.; $\theta = 3 \times 2\pi = 6\pi$; $\mu = 0.4$.

$$\therefore \quad \text{Log } \frac{T_d}{T_s} = .434 \mu \theta.$$

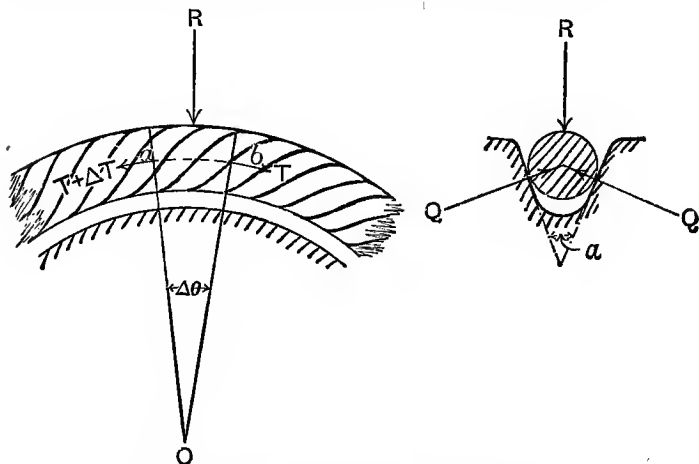
$$\text{,, ,,} = .434 \times .4 \times 6 \times 3.1416$$

$$\text{,, ,,} = 3.27229.$$

$$\therefore \quad \frac{T_d}{T_s} = 1871, \text{ nearly.}$$

$$\text{Or,} \quad T_d = 20 \times 1871 = 37,420 \text{ lbs.}$$

Consequently, a force of 20 lbs. at one end of the rope is able to resist a force of about $16\frac{3}{4}$ tons at the other end.



FRICTION BETWEEN ROPE AND GROOVED PULLEY.

Resistance to Slipping of a Rope on a Grooved Pulley.—The grooves round the rims of the pulleys used for hemp or cotton rope drives are usually V-shaped and of such dimensions that the rope, instead of resting on the bottom of the grooves, gets wedged into them and presses on the sides only. By this means a greater resistance is offered to slipping between the rope and the pulley.

Consider a small length, $a b$, of the rope subtending an angle, $\Delta \theta$, at centre of pulley. Let $T + \Delta T$ and T denote the tensions at a and b respectively. Let Q be the pressure between element $a b$ of rope and sides of groove, α the angle between the sides of the grooves.

Then, from right-hand figure, we get :—

$$R = 2 Q \sin \frac{\alpha}{2}.$$

But, in the previous investigation, we saw that

$$R = T \Delta \theta.$$

$$\therefore 2 Q \sin \frac{\alpha}{2} = T \Delta \theta.$$

Now, the resistance of friction for the element, $a b$, is :—

$$\Delta T = \mu \times 2 Q.$$

$$\therefore \Delta T = \mu \times \operatorname{cosec} \frac{\alpha}{2} \times T \Delta \theta.$$

$$\text{Or, } \frac{\Delta T}{T} = \mu \operatorname{cosec} \frac{\alpha}{2} \Delta \theta.$$

Proceeding to the limit and integrating for the whole arc embraced by the rope on the pulley, we get :—

$$\int_{T_s}^{T_d} \frac{dT}{T} = \mu \operatorname{cosec} \frac{\alpha}{2} \int_0^\theta d\theta.$$

$$\text{i.e., } \log_e \frac{T_d}{T_s} = \mu \theta \operatorname{cosec} \frac{\alpha}{2} \quad \dots \dots \dots \text{(XIV)}$$

Compared with our previous results for a flat pulley, we see that the logarithm of the ratio of the tensions is increased in the proportion of $\operatorname{cosec} \frac{\alpha}{2} : 1$.

Generally, the groove angle is about 45° , for hemp or cotton ropes, and then :—

$$\operatorname{cosec} \frac{\alpha}{2} = \operatorname{cosec} 22\frac{1}{2}^\circ = 2.6.$$

LECTURE VII.—QUESTIONS.

1. Find an expression for the work absorbed in one revolution of an axle on its bearing. An axle is 2 inches in diameter, and the weight pressing it on the bearing is 1,000 lbs., find the number of units of work lost in 100 revolutions. (Take the coefficient of friction as 0.08). *Ans.* 4188.8.
2. Deduce a formula for obtaining (approximately) the work lost in friction in one revolution of a horizontal shaft in its bearing. A horizontal shaft, 9 inches in diameter, is acted on simultaneously by a horizontal force of 3 tons and a vertical force (including its own weight) of 4 tons. Find the horse-power lost in friction when the shaft makes 100 revolutions per minute. The coefficient of friction is 0.07. *Ans.* 5.6 H.P.
3. Explain what is meant by "dead angle" when applied to a steam engine. Explain how you would find it.
4. Explain the nature of "rolling friction," and deduce a formula which approximately represents its amount.
5. What are anti-friction wheels? Find an expression for the work saved by their use over ordinary bearings.
6. Explain, by aid of sketches, how friction is reduced to a minimum in the cases of large crank-shaft bearings and in bicycle bearings.
7. Discuss the advantages and disadvantages of ball bearings. Describe with sketches the construction of an adjustable ball bearing.
8. Write down the formula for finding the work lost in friction in one revolution of a vertical shaft turning on a flat pivot. A vertical shaft 3 inches in diameter, and weighing 30 cwt. (including wheels, &c.), turns on a flat pivot. Find the horse-power lost in friction when the shaft makes 100 revolutions per minute, the coefficient of friction being taken at 0.07. *Ans.* 373 H.P.
9. Explain a formula by means of which the loss by friction on the pivot of a vertical shaft may be calculated, and apply the formula in the following case:—Weight on pivot, 3 tons; diameter of pivot, 4 inches; coefficient of friction, .01; revolutions per minute, 75. Find the horse-power lost in friction. *Ans.* 106.
10. Investigate an expression for the moment of friction of a flat pivot, stating the assumption made as to distribution of pressure. Find the H.P. lost by the friction of a footstep bearing, the diameter of which is 4 inches, the total load on it being 3,000 lbs., the number of revolutions 100 per minute, and coefficient of friction .06. *Ans.* 38.
11. Explain the theory of the anti-friction pivot, and deduce a formula for the work lost in friction in one revolution of the shaft. Draw the curves representing the outline of the pivot.
12. A string is stretched on the circumference of a rough circle; when one of the forces is on the point of preponderance, find the pressure on the circumference, and the tension of the string, at any assigned point.
13. A thread is stretched by two forces, P and Q , over a rough plane curve; when P is on the point of overcoming Q show that $P = Qe^{\mu\theta}$, where θ denotes the angle between the normals at the extreme points where the thread is in contact with the curve. A rope is wrapped three and a-half times round a horizontal cylinder, the coefficient of friction between the rope and the cylinder is $\frac{1}{11}$; a weight of 1,000 lbs. is fastened to one end of the rope, what weight must be fastened to the other end to prevent sliding, the weight of the rope being neglected? Take $e = 2.72$. *Ans.* $Q = P/e^2 = 135.14$ lbs.

14. Deduce a formula for the resistance offered to slipping of a rope round a grooved pulley in terms of groove angle, angle subtended at centre of pulley by the rope and coefficient of friction. Find the maximum ratio of tensions in the tight and slack ends of a rope passing over a grooved pulley, the angle of the groove being 40° , coefficient of friction $\cdot 15$, arc of pulley embraced by rope $\frac{3}{4}$ of circumference. *Ans.* $5\cdot22$.

15. Establish a formula giving the ratio between the tensions at the extremities of a rope which is coiled through a given angle round a post. A rope has its direction changed through two right angles by passing round a grooved guide pulley whose diameter is 12 inches, the diameter of the axle of the pulley being $1\frac{1}{2}$ inches, and the coefficient of axle friction being $0\cdot07$. How is the efficiency of the pulley affected by axle friction when a load of 2,500 lbs. is being raised? If the pulley was fixed so that it could not turn, how would the efficiency be affected by the friction of the rope on the pulley, taking the coefficient as $0\cdot6$? *Ans.* Efficiency = $98\cdot4$ per cent.; efficiency = $15\cdot2$ per cent.

16. Describe, with sketches, a good ball bearing of any kind. State clearly the nature of rolling resistance. What is known about its amount, experimentally? What are the advantages and disadvantages attending the use of such bearings?

17. Sketch and describe the construction of either the hub of a cycle wheel or of any other ball bearing with which you are acquainted. Explain the advantages and disadvantages of ball bearings over the ordinary axle or shaft bearing, stating the reasons for your answers. (S. and A., Adv.)

18. A shaft makes 50 revolutions per minute. If the load on the journals is 8 tons and the diameter 7 inches, at what rate is heat being generated, the average coefficient of friction being $0\cdot05$? Imagine the bearing to be a slack fit on the journal. *Ans.* $106\cdot1$ B.T.U. per minute. (B. of E., Adv., 1900.)

19. A direct-acting engine has a piston diameter of 16 inches, a stroke of 2 feet, and a connecting-rod 4 feet long. Find the turning moment on the crank shaft, due to steam pressure, when the piston has moved through one-quarter of the stroke from the back end, the effective steam pressure on the piston being then 50 lbs. per square inch. Graphical constructions may be used. (C. & G., 1902, O., Sec. A.) *Ans.* $9,219$ lb.-feet.

20. The figure shows a bent lever, AOB. The fulcrum at O is in a loose cylindric bearing 4 inches diameter, coefficient of friction $0\cdot3$. AO is 12 inches, BO is 24 inches; the force Q of 1,000 lbs. acts at A. What



force P acting at B will just overcome Q and the friction? Find also the line of action and magnitude of the force acting upon the lever at O. (B. of E., S. 2 and 3, 1904.)

N.B.—See Appendices B and C for other questions and answers.

LECTURE VII.—A.M. INST. C. E. EXAM. QUESTIONS.

1. What are the laws of friction between solid bodies and between a solid and liquid. In the case of a journal immersed in oil, what law would apply? (I.C.E., *Feb.*, 1900.)

2. Find the H.P. absorbed in overcoming the friction of a footstep-bearing 4 inches diameter, the total load being $1\frac{1}{2}$ ton, the number of revolutions being 100 per minute, and the average coefficient of friction 0.07. (I.C.E., *Oct.*, 1900.)

3. A crank-shaft, diameter $12\frac{1}{2}$ inches, weighs 12 tons, and it is pressed against the bearings by a force of 36 tons horizontally. Find the horse-power lost in friction at 90 revolutions per minute. Coefficient of friction = 0.06. (I.C.E., *Oct.*, 1902.)

4. Find the conditions of limiting equilibrium of a wheel and axle when the friction of the axle in its bearings is taken into account. If a , b , c are the radii of the wheel, the axle, and the bearings, and α the angle of friction, and, if $c \sin \alpha$ is small, show that about

$$c \frac{(a+b)}{ab} \sin \alpha$$

of the work done in raising the weight is wasted. (I.C.E., *Feb.*, 1903.)

N.B.—See *Appendices B and C* for other questions and answers.

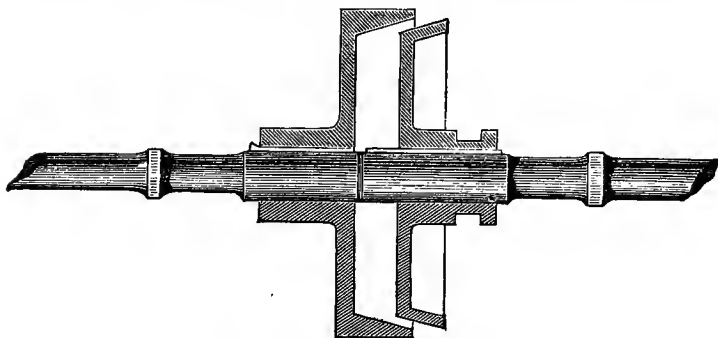
LECTURE VIII.

CONTENTS.—Friction Usefully Applied—Friction Clutches—Frustum, Addyman's, and Bagshaw's—Weston's Friction Coupling and Brake—Grooved Disc Friction Coupling—Weston's Centrifugal Friction Pulley—Brakes Defined and Classified—Block Brakes—Flexible Brakes—Proper Rotation of Brake-Wheel when Lowering a Load—Mathematical Proof—Example I.—Paying-out Brake for Submarine Telegraph Cables—Differential Brake for Lord Kelvin's Deep-Sea Sounding Machine—Dynamometers—Absorption Dynamometers—Prony Brake—Method of Taking Test for Brake Horse-Power—Example II.—Improved Prony Brake—Appold's Compensating Lever—Semicircular Strap Dynamometer—Society of Arts Rope Dynamometer—Advantages of Rope Brake—Tests of Engines with Rope Brake—Transmission Dynamometers—von Hefner-Alteneck or Siemens'—Rotatory Dynamometers—Epicyclic Train; King's, White's, or Webber's—Spring Dynamometers—Ayrton and Perry's and van Winkle's—Hydraulic Transmission Dynamometers—Flather's and Cross'—Tension Dynamometer for Submarine Cables—Froude's Water Dynamometer—Questions.

Friction Usefully Applied.*—As we remarked in Lecture X. of our *Elementary Manual of Applied Mechanics*, friction has its advantages as well as its disadvantages. And, although it is the duty of the engineer to reduce friction to a minimum in the case of the bearings of engines, shafting, and machines in general; yet, he has, nevertheless, to devise means for producing a maximum of friction in the case of brakes, blocks, and grips whereby motion has to be arrested gradually or suddenly; or, in the case of friction gearing, pulleys, and clutches whereby power has to be transmitted from one shaft to another. Or, he may have to arrange for a more or less constant retarding force as in the case of absorption dynamometers when used for paying-out submarine cables or for determining the brake-horse power of motors. It will, therefore, be our endeavour, in this Lecture, to describe these various methods of usefully employing friction with suitable illustrations applicable to each case.

* The student should refer to Thomas W. Barber's *Engineers' Sketch Book of Mechanical Movements*, published by E. & F. Spon, London, for a large variety of Brakes and Retarding Appliances in Section 5, Friction Clutches in Section 15, and Friction Gear in Section 38, and to an excellent book on *Dynamometers and the Measurement of Power*, by John F. Flather, Ph.B., published by John Wiley & Sons of New York, as well as to the several papers referred to by footnotes in this Lecture.

Friction Clutches.*—When two light shafts are in line with each other, one of which has to be set in motion or stopped at any time, whilst the other one is kept rotating, they may be conveniently coupled together by the simple form of clutch illustrated by the accompanying figure. To the left-hand shaft there is keyed



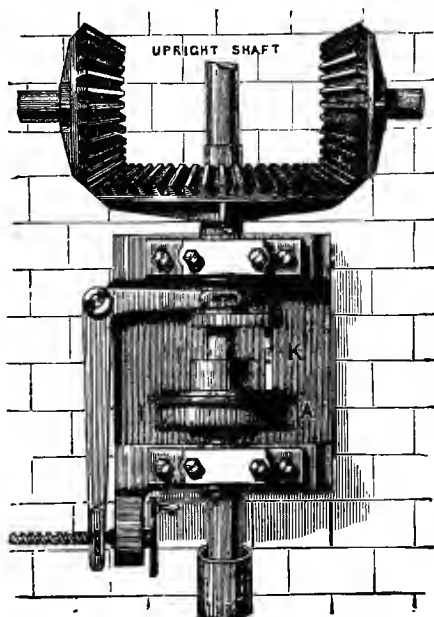
SIMPLE FRUSTUM FRICTION CLUTCH.

a hollow disc, truly bored out to a certain taper, which depends upon the materials, and must be great enough to prevent jamming. On the right-hand shaft another disc fits freely on a feather-key, so that it may be forced to the right or left by a forked lever (not shown) which engages the groove in its boss. This right-hand disc is turned to the same taper as the left-hand one; consequently, when it is pressed home thereon and held in position by a locked or weighted lever, the friction between the two conical surfaces is sufficient to transmit power from the left-hand to the right-hand shaft. In order that the shafts may remain in line, each disc is supported by a bearing immediately behind its own boss.

The following figure shows how this same sort of clutch might be applied to a vertical shaft gearing with, and transmitting motion to, one or two horizontal shafts. Here the lower end of the bell-crank lever is forked and fitted with a nut which engages a screw worked by a wheel and handle. The upper end of the bell-crank is also forked, and should engage the grooved collar of

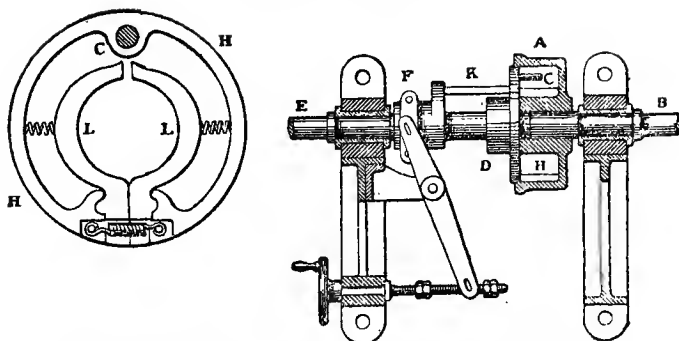
* We are indebted to Messrs. J. Bagshaw & Sons, Batley, for three of the following figures, and to the Council of the Institution of Civil Engineers for the third and fourth figures, which are taken from Mr. Walter Bagshaw's paper on "Friction Clutches." See *Proc. Inst. C.E.* for 1886-87, vol. lxxxviii., p. 368.

the movable part of the clutch. The makers, however, prefer the method shown in the next and following illustrations, whereby the wedge-pointed rod, K, on being depressed, forces open two internal levers, L L, which, in turn, expand a split friction ring, H, until it grips the inside of the hollow clutch, A.



ADDYMAN'S FRICTION COUPLING FOR UPRIGHT SHAFTING.

Another method by the same firm is shown in the second figure on the following page. Here, one internal forked lever, D (which clears the shaft), is fixed at its lower end to a right- and left-handed screw. When this lever is forced forward the screws are turned in one direction, and press out the split cast-iron ring, A, until it bites the internal surface of the hollow clutch, and starts the machine connected therewith. On the other hand, when the lever is pulled back, the screws are turned in the reverse direction and pull the ends of the split-iron ring together, thus freeing it from the clutch and permitting the driven machine to come to a stand-still.

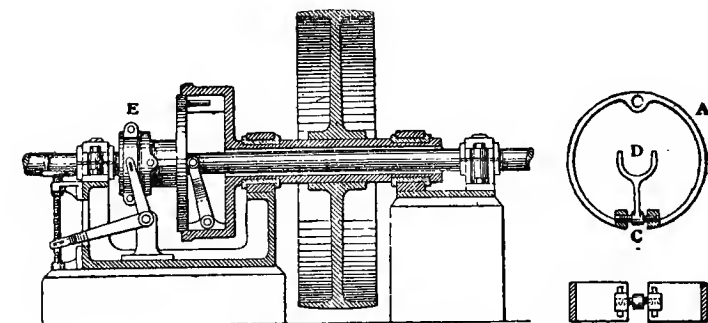


ADDYMAN'S FRICION COUPLING FOR HORIZONTAL SHAFTING.

INDEX TO PARTS.

A for Hollow shell of clutch.
 B „ Driving shaft.
 C „ Crank pin or driver.
 D „ Disc.
 E „ Driven shaft.

F for Collar.
 H „ Friction ring.
 K „ Wedge-pointed rod.
 L „ Levers.

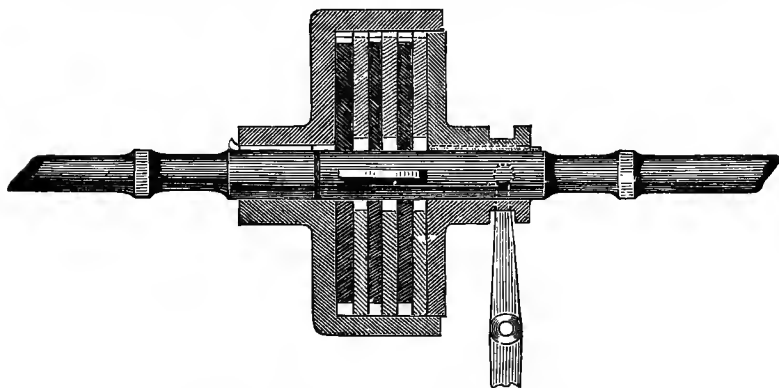


BAGSHAW'S HOLLOW SLEEVE CLUTCH.

Weston's Friction Coupling and Brake.—This apparatus consists of a shell keyed on one of the shafts, in which are fitted two series of friction discs free to slide lengthwise towards or from each other. The series, shaded black in the figure, is made to rotate with the right-hand shaft by means of feather-keys, and the other series (shaded light) with the outside shell. When no compression is applied to the discs through the lever, the shafts are free to revolve independently of each other, but upon com-

pressing the discs, the rotary motion of one series is transmitted wholly to the other, and the apparatus acts as a coupling. When the pressure is slightly relieved, the friction between the two sets of discs acts as a brake and thus controls the speed.

It will be observed that this coupling is designed on the principle of multiple-gripping surfaces, and, as friction increases in proportion to the number of pairs of surfaces in contact, it is possible to so increase their number and the extent of the surface in contact, so as to multiply the resistance due to friction to any desired amount. Weston's arrangements of alternate discs of iron or steel and gun-metal, or metal and wood or leather, are used for



WESTON'S FRICTION COUPLING AND BRAKE.

a variety of devices, such as lowering and holding brakes in large cranes, elevating gear in large guns, &c. The resistance obtained in this way is very remarkable. It is stated by Prof. Goodeve in his *Principles of Mechanics*, that "six discs of iron, $14\frac{1}{2}$ inches in diameter, riding between wooden discs and used in a windlass, are recorded to have sustained a direct pull on the cable of 34 tons without yielding."

Concentric Grooved Disc Friction Coupling.—Another form of friction coupling is that known as "Robertson's Wedge and Groove Friction Clutch," which will at once be understood from the accompanying sectional plan and elevation. If the discs, D_1 , D_2 , forming the two parts of the coupling, are to be brought into contact, then the lever, L , is moved to the right, which turns the forked claw or clip, CL , connected to the central circular pin, CP , inside the eccentric boss, EB , of the bearing, B . This claw in turn forces forward the collar, O (along the feather of the shaft),

connected to the disc, D_1 , and thus brings D_1 into gear with D_2 . This latter disc is fixed to the pulley or wheel which is connected to the machine to be set in motion. To bring the discs out of gear the lever, L , is moved to the left, and precisely the reverse action takes place. The advantage of all these several forms of friction couplings is, that they transmit power without jar and will slip under an excess of force or shock beyond that which they are designed to transmit. The forces transmitted are, however, limited by the coefficient of friction and the number, extent, and exact fit and freedom from oil or moisture of the surfaces in contact. We shall have to refer to spur and bevel frictional gearing later on, and to investigate the limits to which such appliances may be adapted.

Weston's Centrifugal Friction Pulley.*

—Two forms of this friction pulley are shown in the following figures, the first suitable for rope and the second for belt driving. This pulley is specially well adapted for driving high speed machines, such as centrifugals and dynamos.

The principal advantages claimed for it are :—(1) A number of machines may be driven direct from the same shaft; and any one of them may be started or stopped independently of the others.

(2) Both starting and stopping are performed gradually and easily.

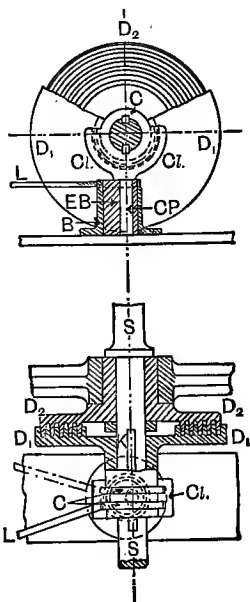
(3) No sudden shocks or stresses are caused to the pulley, the belt, or any part of the driven machine, since the necessary friction for imparting the motion is applied automatically by centrifugal action.

(4) When the friction pulley is the driver no loose pulleys are required, and consequently there is no wear and tear of belting from shifting forks, &c.

(5) The pulley may be so adjusted as *not* to transmit more than the desired power without slipping or giving warning that an extra load has been brought into circuit.

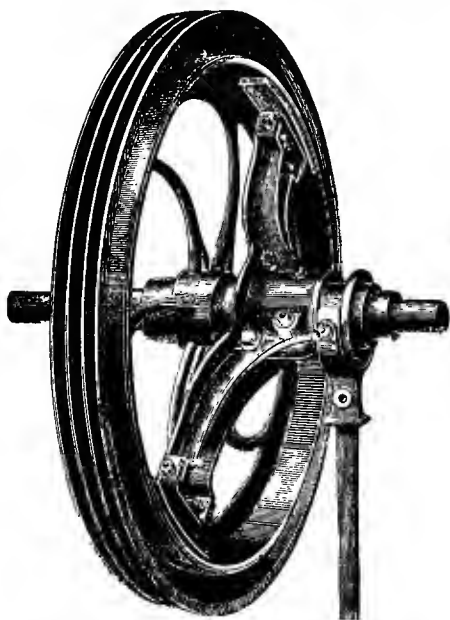
The construction and action of this pulley will be readily

* We are indebted to Messrs. Watson, Laidlaw & Co., of Glasgow, the makers of this friction pulley, for the two following illustrations.

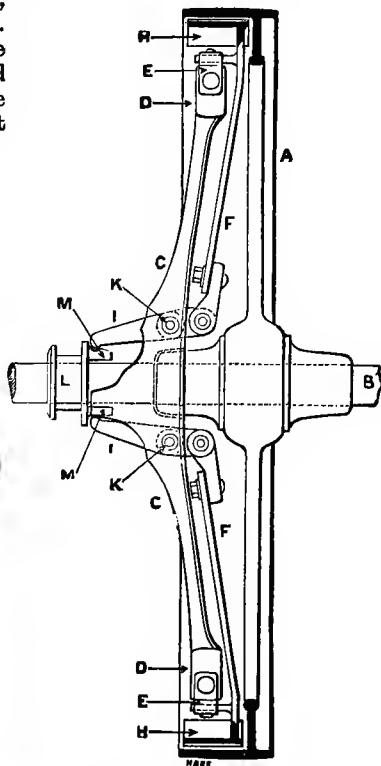


CONCENTRIC GROOVED
DISC FRICTION COUPLING.

understood by reference to the following figures. A, is the pulley proper, which carries the belt, and rides loose on the shaft B. C, is an arm bolted or keyed to the shaft, and revolves with it. This arm carries two toe-levers, I, pivoted at K. To these toe-levers are attached the friction arms, F, the latter being again connected to the arm, C, at the ends, D, by means of flexible springs, E. The gripping surfaces, H, of the arms, F, are faced with leather, and turned to the same curvature as the inside of the rim of the pulley. It



FRICITION PULLEY FOR ROPE DRIVE.



VERTICAL SECTION OF FRICITION PULLEY FOR BELT DRIVE.

will thus be seen that when the arm, C, is in motion it will carry round with it the two friction arms, F. The latter will tend, under the influence of centrifugal action, to fly outwards, and thus to bind themselves against the rim of the pulley, A, and carry the latter round with them. In the condition shown in the illustration, this tendency is restrained by the toes, M, of the

sliding sleeve, L. All that is necessary, therefore, in order to permit the friction arms to transmit their motion to the pulley is to withdraw the toes, M, by moving the sleeve, L, a very short distance along the shaft by means of a hand lever, such as is shown in the left-hand figure. The springs, E, prevent the motion of the friction arms being too suddenly communicated to the pulley; hence, as we said before, there is no sudden stress put upon the belt, and it gradually acquires its full speed.

In order to get the best and most economical results, the highest convenient speed should be arranged for this friction pulley.*

Brakes Defined and Classified.—The contrivances comprised under the general title of brakes, are those by means of which friction is intentionally opposed to the motion of a machine, in order to stop it, retard it, or employ some of its superfluous energy with the view of producing uniform motion.

Brakes may be classified as follows :—

1. *Block brakes* are those in which one solid body is simply pressed, and rubs, against another.

2. *Strap, or flexible, brakes* are those which embrace the periphery of a drum or pulley.

3. *Pump brakes* are those in which the friction amongst the particles of a fluid produces resistance when the fluid is forced through restricted passages.

4. *Turbine, or fan, brakes* are those in which the resistance employed is that of a fluid to a fan rotating in it.

Block Brakes.—The most familiar example of the use of the ordinary block brake is its application to road vehicles and railway rolling stock. Its effect is to retard or stop the rotation of the wheels, and thus to make them slip instead of rolling on the road or railway. The resistance caused by such a brake to the motion of a carriage may be less than, but can never be greater than, the friction between the stopped wheel and the road or rail.

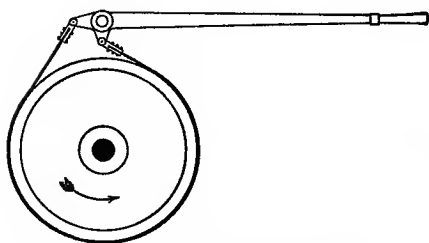
Flexible Brakes.—In hoisting and lowering machinery, such as crabs, winches, cranes, and colliery winding engines, a form of brake called the friction strap or flat brake is generally employed for holding the load when raised to the desired height, or for gradually arresting its motion on being lowered to the required

* There are an immense number of patented "Friction Power Transmitters," but we have given sufficient illustrations to show the application of these useful devices. The student should, however, consult the pages of *The Engineer, Engineering*, and other similar periodicals—e.g., he should refer to *The Engineer* of April 18, 1890, for a description and complete set of sectional figures of Shaw's "Coil Friction Power Transmitter," which is an interesting departure from the usual methods of bringing machines into circuit with their drivers.

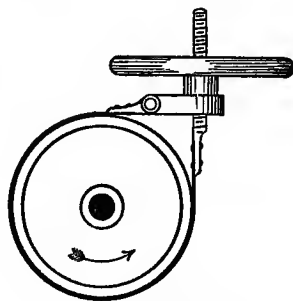
depth. In the case of ordinary small crabs, winches, and cranes, this brake takes the form of a flexible steel strap, which, to a greater or less extent, encircles a strong flat-faced cast-iron wheel or pulley. It is made of sufficient length to clear the wheel when slack. The strap is tightened by means of a lever actuated by the hand or foot. If the load to be arrested is comparatively small, then the brake-strap may be conveniently fixed to the barrel-shaft of the crab or crane, and it naturally takes the form shown by the left-hand figure, where the inner end of the lever terminates in a bell-crank.*

The extremities of the arms of this bell-crank are attached to the ends of the brake-strap. This brake-strap is simple, can be readily fitted to most hoisting gear, and possesses great gripping force.

In double or treble purchase crabs and cranes where the load to



SIMPLE BRAKE-STRAP AND LEVER.



SIMPLE BRAKE-STRAP
AND SCREW.

be held in position or arrested is greater than in the case of single purchase ones, the brake is usually fixed to the second or third motion shaft, and may, for convenience of manipulation, take the form shown by the right-hand figure. As in the previous case, the brake-wheel is usually made of cast-iron, with a solid central web between its boss and rim. The upper end of the thin flexible steel strap is fixed to a projecting arm or stretcher, and the other end to the lower terminal of a vertical screw.

This screw is raised or lowered by a horizontal hand-wheel whose boss is a nut fitting the vertical screw. When lowering a load rapidly by means of a brake, it is usual to throw the first

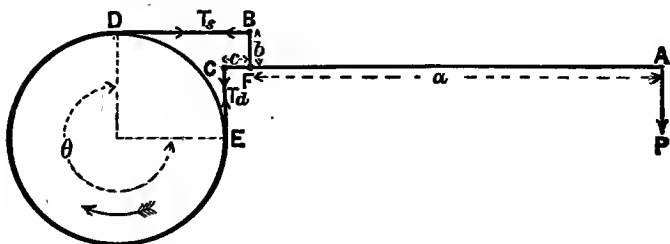
* Also, see the figures in Lecture XIII. of the author's *Elementary Manual of Applied Mechanics*, as well as the figures of the Crab in Lecture IX. of Vol. V., on *Theory of Machines*, of this Text-book.

motion shaft out of gear by aid of a pawl, &c., as explained in Lecture XIII. of our *Elementary Manual*.*

Proper Direction of Rotation of Brake-Wheel when Lowering a Load.—In fixing a brake-strap to any piece of hoisting machinery, care should be taken that the brake-wheel turns (as shown by the arrows on the two previous figures) in a direction so as to produce the greater stress upon that end of the brake-strap which is attached to the fixed end of the lever or projecting arm; for, if fitted in the reverse order, it will be found much more difficult to control the motion of the load.

Mathematical Proof of the Previous Statement.—The truth of the previous statement may be proved by calculating the frictional moments about the centre of the brake-wheel.

Let T_d and T_s denote the tensions in the tight and slack ends of the brake-strap. Which is the tight and which the slack end will, of course, depend upon the direction of rotation of the brake-wheel. In the accompanying figure the direction of rotation is



GEOMETRICAL DIAGRAM OF BRAKE-WHEEL, STRAP, AND LEVER.

indicated by a curved arrow; and, consequently, CE is the tight and BD the slack end.

- Let θ = Angle subtended by strap at centre of wheel.
 „ a, b, c = Lengths of arms, AF, BF, CF respectively.
 „ r = Radius of brake-wheel.
 „ P = Force exerted at end, A, of the lever, AC.
 „ μ = Coefficient of friction between strap and wheel.
 „ M = Frictional moment about centre of brake-wheel.

$$\text{Then,} \quad M = (T_d - T_s) r. \quad \dots \dots (1)$$

$$\text{But, as we proved in} \quad \left\{ \begin{array}{l} \text{equation (XI), Lec-} \\ \text{ture VII.,} \end{array} \right. \quad \frac{T_d}{T_s} = e^{\mu \theta} = k. \quad \dots \dots (2)$$

* See *Notes on the Construction of Cranes and Lifting Machinery*, by Edward C. R. Marks. Published by John Heywood, Deansgate, Manchester, first edition, pp. 78 to 81, and Fig. 83, for an improved form of lowering brake-strap, which is lined with leather and fitted to the third motion shaft of the crane gear.

Taking moments about F, the fulcrum of the lever, we get :—

$$P a = T_s b + T_d c. \quad (3)$$

By eliminating T_d and T_s from these three equations, we can express M in terms of P , a , b , c , k , and r , all of which have known values.

Thus, from (2), $T_d = k T_s$.

Substituting this value of T_d in (1) and (3), we get :—

$$M = (k-1) r T_s. \quad (4)$$

And, $P a = (b + c k) T_s. \quad (5)$

Dividing (4) by (5), $\frac{M}{P a} = \frac{(k-1) r}{b + c k},$

$$\therefore M = \frac{P a r (k-1)}{b + c k}. \quad (XV)^*$$

Very often the *tight* end, C, of the strap is immovable, for it may be fixed by a pin to the fulcrum, F. In that case $c = 0$, and we have :—

$$M = \frac{P a r (k-1)}{b}. \quad (XV_a)$$

If, however, the direction of rotation be the opposite of that given in the figure, then BD becomes the tight end of the strap, and it is easily proved that :—

$$M = \frac{P a r (k-1)}{b k}. \quad (XV_b)$$

Now, k is always greater than unity, hence we see from (XV_a) and (XV_b) that the resistance to friction is less for one direction of motion than for the other. An example will make this point still clearer.

EXAMPLE I — A treble-purchase crab is fitted with a strap friction-brake worked by a lever. The shaft on which the brake-wheel is keyed, carries a pinion of 12 teeth which gears with a wheel of 48 teeth on the next shaft. This second shaft has a pinion of 12 teeth gearing with another wheel of 60 teeth on the drum or barrel shaft. The diameter of the drum or barrel is 14 inches; diameter of brake-pulley $2\frac{1}{2}$ feet; length of brake handle 3 feet. One end of the brake-strap is immovable, the other end being fixed to the shorter arm of the brake-lever, which is 3 inches long. The angle subtended by the strap at centre of brake-pulley is 270° . If a force of 50 lbs. be applied at the end of the brake-lever, what is the greatest load for each direction of

* The Roman Numbers for these equations follow those in Lecture VII.

rotation of the brake-wheel which could be supported on the end of the rope that passes round the drum? Take $\mu = \cdot 1$.

ANSWER.—The friction moment at the brake-pulley is:—

$$M = \frac{P a r (k - 1)}{b} \dots \dots \dots (1)$$

$$\text{Or, } M = \frac{P a r (k - 1)}{b k} \text{ (according to the direction of rotation) } (2)$$

$$\text{Since, } \theta = 270^\circ = \frac{3\pi}{2}; \text{ and, } k = e^{\mu \theta}.$$

$$\therefore \log k = \cdot 4343 \mu \theta = \cdot 4343 \times \cdot 1 \times \frac{3 \times 3 \cdot 1416}{2} = \cdot 20466.$$

$$\therefore k = 1 \cdot 602, \text{ nearly.}$$

Substituting $P = 50$ lbs.; $a = 36$ ins.; $b = 3$ ins.; $r = 15$ ins. in (1), we get:—

$$M = \frac{50 \times 36 \times 15 (1 \cdot 602 - 1)}{3} = 5,418 \text{ in.-lbs. } (3)$$

$$\text{Or, from (2), } M = \frac{50 \times 36 \times 15 (1 \cdot 602 - 1)}{3 \times 1 \cdot 602} = 3,382 \text{ in.-lbs. } (4)$$

Now, the moment of the couple at the brake-wheel, due to the load, W , at the drum, is:—

$$M = W \times 7 \times \frac{12}{60} \times \frac{12}{48} = \frac{7W}{20} \text{ in.-lbs.}$$

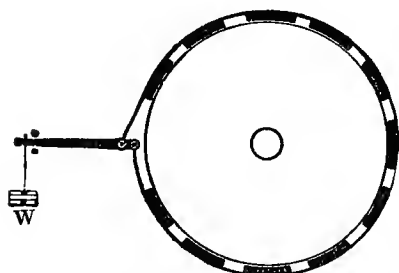
$$\text{Hence, from (3), } W = \frac{5418 \times 20}{7} = 15,480 \text{ lbs.}$$

$$\text{Or, from (4), } W = \frac{3382 \times 20}{7} = 9,663 \text{ lbs.}$$

This example shows at once the importance of attending to the direction of rotation of the brake-wheel when the load is being lowered, before fitting up the brake appliance. The rule is, therefore, *to make the tight end of the strap the fixed or immovable end, and to attach the slack end to the shorter arm of the brake lever*. By an inspection of the arrangement, it becomes evident that for one direction of rotation of a brake-wheel, the friction between the strap and its pulley assists the effort on the lever, whilst it opposes it for the reverse direction.

Paying-Out Brake for Submarine Cables.—When a brake-strap exceeds 2 or 3 feet in diameter, it is usually fitted throughout its inner surface with wooden blocks—preferably of hornbeam or beach. These are screwed to the steel strap from the outside thereof. The “paying-out gear” for the laying of submarine

cables (see accompanying illustration) always contains one or two of these larger brakes of from 6 to 8 feet in diameter. The cable as it comes from the tanks of the ship is coiled four or five times round the paying-out drum, from which it then passes aft under the dynamometer pulley and over the stern sheaves into the sea. Now, a restraining force must be applied to the cable in order that it may be laid as evenly as possible along the irregular bed of the ocean, with just the desired amount of slack, so as not to put too great an initial stress thereon, and to permit of the cable being lifted for future repairs, without having recourse to cutting the same, when in moderate depths say up to 1,000 fathoms. This restraining force cannot be directly put upon the cable without injuring it, so recourse is had to the device of fixing one or two large brakes, of the Appold type, to an extension of the paying-out drum shaft. The accompanying figure will serve to indicate to the student the kind of brake generally employed for this purpose. The engineer in charge calculates the necessary stress required for the particular type of cable, depth of water, and speed of ship; and after making his calculations he applies the desired weights, W , at the end of the brake-lever.



APPOLD'S BRAKE FOR THE PAY-OUT GEAR
OF SUBMARINE CABLES.

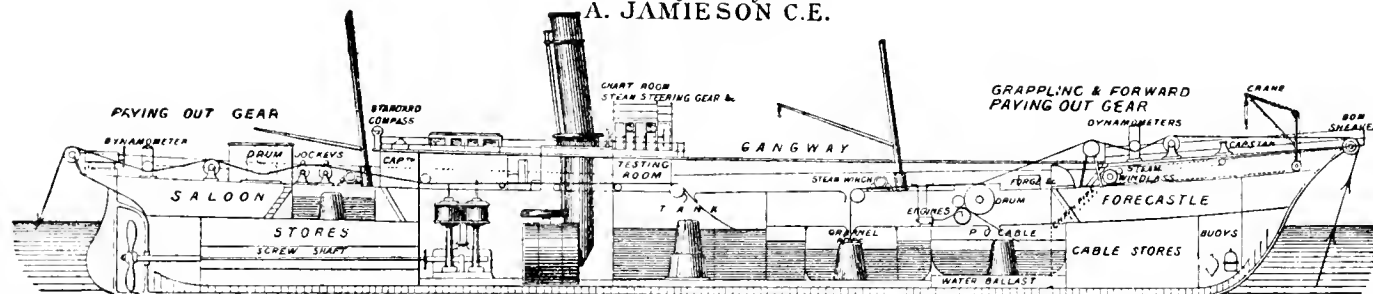
The brake runs in a trough of water, so that the heat generated between the wooden blocks and the brake-wheel may be carried off quickly, as well as to ensure that the coefficient of friction may remain as constant as possible. In order to test whether too much or too little slack is being paid out at any time, the engineer adds a slight excess of weight over his

calculated amount, W , for a short time, and then subtracts a slight amount for an equivalent time; when, by aid of the tachometer or speed counter on the paying-out drum, and the ship's log, as well as the known length of cable for each revolution of the drum, he is able to permanently adjust the required amount of brake-weight during a run of several nauts for a uniform speed of the ship.

Differential Brake for Lord Kelvin's Deep-Sea Sounding Machine.—Another interesting illustration of the application of a brake to marine purposes is contained in Lord Kelvin's deep-sea pianoforte-wire sounding machine. The object to be attained by

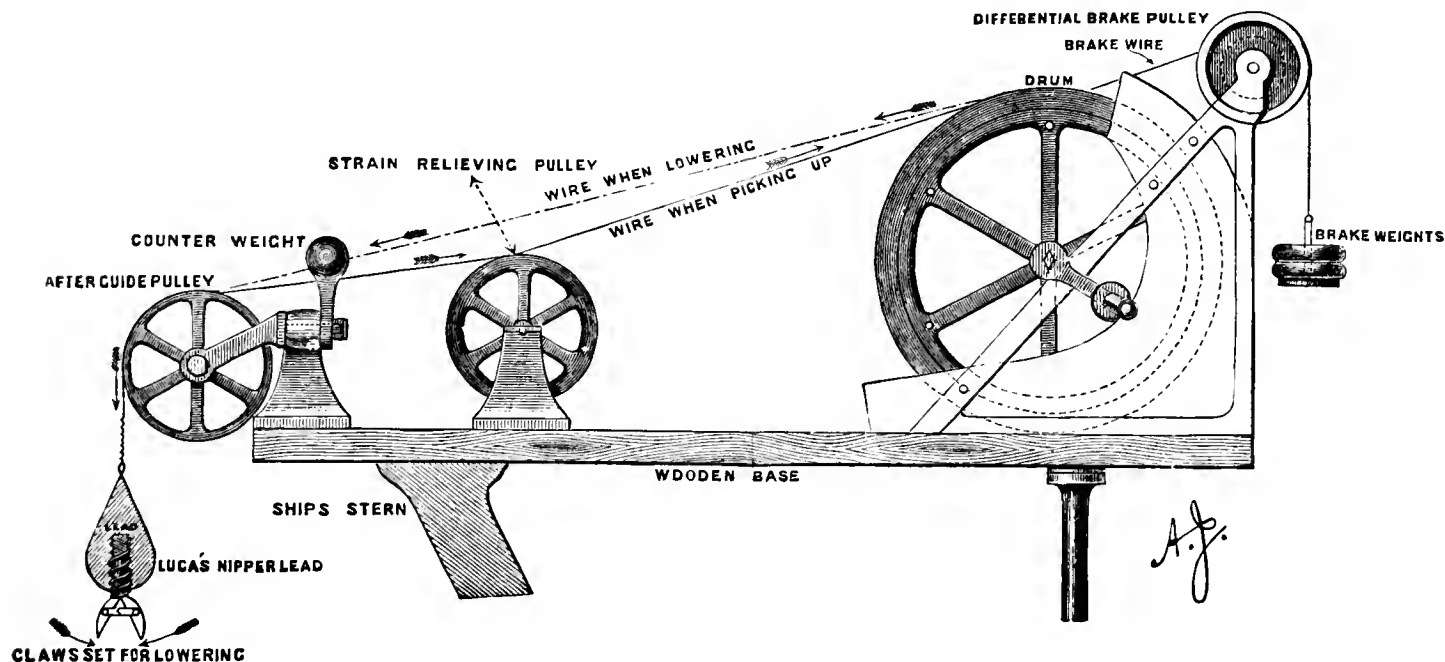
TELEGRAPH SHIP

Designed by
A. JAMIESON C.E.



POSITIONS OF PAYING OUT AND PICKING UP DRUMS WITH THEIR FRICTION BRAKES AND THE DYNAMOMETERS, &C.,
ON A TELEGRAPH SHIP.

For the other views of this Ship and a complete description thereof, see "The Telegraphic Journal" of Aug. 15, 1881.



LORD KELVIN'S DEEP-SEA PIANOFORTE-WIRE SOUNDING-MACHINE WITH DIFFERENTIAL BRAKE-PULLEY.

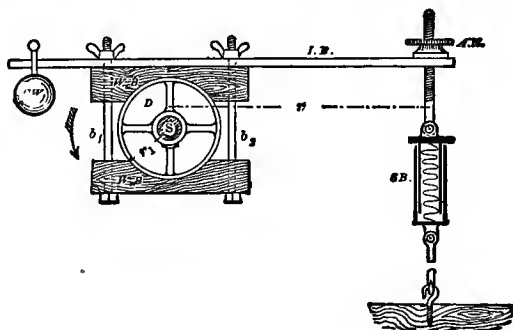
this differential brake is to put the necessary restraining force upon the sounding wire as it is being lowered, and to suddenly stop the same immediately the lead reaches the bottom of the ocean, so that the correct depth may be registered by the counter attached to the wire-drum. The brake weights are attached to a rope which passes over, and is fixed to the larger pulley (shown at the right-hand top corner of the figure), whilst the brake-wire is attached to a smaller pulley on the same spindle. This brake-wire passes round a V-groove on the side of the drum containing the pianoforte wire, and its other end is fastened to the framing. When heaving up the pianoforte wire from a great depth with the "lead" attached thereto, it was found necessary to take a turn of the sounding-wire round the "strain-relieving pulley" in order to prevent the drum being damaged by the constant tension on the sounding-wire.

Dynamometers.*—It is frequently of great practical importance to ascertain by direct experiment the "nett" power developed by motors or expended in driving machines. For example, steam, gas, and oil engines, turbines, water wheels, and electric motors, &c., are being designed, made, and sold every day to drive machinery of one kind or another. It is, therefore, surely far better, both for the buyer and the seller, to know the *exact* "Brake horse-power" (B.H.P.) which a generator will develop at a certain speed and under a certain mean pressure, than to vaguely talk of the "Nominal horse-power" (N.H.P. whatever that may mean); or even to speak of the "Indicated horse-power" (I.H.P.) in the case of engines; or the "Gross horse-power" in regard to hydraulic machines; or the "Electrical horse-power" (E.H.P.) with respect to dynamos. It is also surely far better to know exactly the Brake horse-power which a certain machine, or even a section or the whole of a factory requires when worked under certain conditions, than to make a rough guess at the amount from vague data or even previous general experience. For, it is by aid of such tests that the "mechanical efficiency" of the motors and machines can be accurately determined and further improvements take place in raising their efficiency.

It is, therefore, with a certain pleasure that the author places before the student a few of the many devices that have been invented for obtaining the brake horse-power developed by motors or required for driving machinery; because, he has been persistently advocating the adoption of this system of gauging power for many years. Ever since the introduction of electric lighting and the transmission of power by electricity, this view of the case

* The word dynamometer is derived from the Greek words *δύναμις*, signifying *force*, and *μετρεω*, to *measure*.

has year by year become more appreciated and been taken advantage of. It is not too much to say, that the general engineer has been greatly indebted in this respect, to his colleague the electrical engineer. For all powers up to 200 H.P. or so, there is no practical difficulty in making exact brake trials; and probably in the near future, we shall see engines of 1,000 H.P. and upwards tested and paid for by this uniform and reliable standard. There are two main types of mechanical dynamometers—(1) Absorption or Friction Brake; (2) Transmission. Absorption dynamometers absorb the work which they help to measure and dissipate the same as heat. Whereas, transmission dynamometers pass on the work which they help to measure, and only waste a small



ORIGINAL PRONY BRAKE DYNAMOMETER.

INDEX TO PARTS.

W B for Wooden blocks.	IB for Stiff iron bar.
D „ Drum or pulley.	SB „ Salter's balance.
S „ Driving shaft.	CW „ Counter weight.
$b_1 b_2$ „ Bolts with ram's horn nuts.	AN „ Adjusting nut.

fraction of the total work delivered to them. In the first instance, we shall describe with an example the Prony brake, because it will form an easy introduction to more recent and perfect kinds.

Absorption Dynamometers.—Prony Brake.—This dynamometer is only a particular application of the friction brake already mentioned in this Lecture. As will be seen from the following figure and “index to parts,” a pulley or drum, D, keyed to the shaft, S, is gripped between two wooden blocks, WB, which may be tightened or loosened (as required) to produce more or less friction between them and the pulley) by turning the ram's horn nuts of the bolts, $b_1 b_2$. An iron bar fixed across the top of the upper block (or, if preferred, along the bottom of the lower one) is balanced by a counter weight, CW, and has either an adjustable

weight or a Salter's spring-balance, S B, fixed to the other end at a known distance or radius, r , from the centre of the shaft. Or, the counter weight may be dispensed with, and the bar extended to the left side instead of to the right, and allowed to press upon the platform or table of an ordinary Pooley weighing machine.*

Method of Taking Test for Brake Horse-Power. — 1. Adjust position of C W until it balances the weight of I B, A N, and S B, with the wooden blocks slack on the pulley.

2. Start machinery and tighten blocks, W B, by the ram nuts until the desired speed is attained. At the same time, adjust S B by nut, A N, until a balance is obtained, taking care to keep I B level by aid of a length-rod or pointer.

3. Note number of revolutions per minute by a tachometer or speed indicator if great, or a counter and stop-watch if slow.

4. Note the stress indicated by spring balance.

Then, the horse-power absorbed by the brake is obtained from the formula :—

$$\text{B.H.P.} = \frac{2 \pi r n P}{33,000}.$$

Where, r = Radius or horizontal distance from centre of balance to centre of shaft in feet.

„ n = Number of revolutions per minute.

„ P = Pull indicated by the Salter's balance.

Since, $\frac{2 \pi}{33,000} = .0001904 = \text{a constant.}$

We get, $\text{B.H.P.} = .0001904 \times r \times n \times P.$

EXAMPLE II.—A small fast-speed Westinghouse engine was fitted with a Prony brake of the form just described. The fly-wheel was 2 feet diameter and 6 inches broad. The horizontal distance from the centre of the crank-shaft to the centre of the spring-balance was 2.5 feet; the mean revolutions per minute

* This latter method of registering the forces produced by the friction between the revolving pulley and the stationary wooden blocks is very handy, when engines are having their steam consumption registered for long continuous runs, during the process of getting their bearing surfaces into good working condition. Any alteration in the balancing force is easily effected by simply shifting the small adjusting weight along the light arm of the weighing machine, and little or no attention need be paid to the brake lever. Besides which, there can be no danger to the attendant in the case of the pulley firing and seizing the wooden blocks.

were 624, and the mean pull on the spring-balance was 48 lbs. Find the brake horse-power.

Here, $r = 2.5$ ft.; $n = 624$; $P = 48$ lbs.	Or, by logarithms :—
$\therefore \text{H.P.} = .0001904 \times r \times n \times P$	log. $.0001904 = \bar{4}.2797$
$\therefore \text{H.P.} = .0001904 \times 2.5 \times 624 \times 48$	" $2.5 = 0.3979$
	" $624 = 2.7952$
	" $48 = 1.6812$
$\therefore \text{H.P.} = 14.26.$	Antilog. of $1.1540 = 14.26.$

It is important to note, that neither the diameter of the pulley nor the pressure of the friction blocks on the same (due to the weight of the apparatus or the tightening of the ram nuts), nor the coefficient of friction enter into the formula for obtaining the horse-power. The only data required being the horizontal length of lever, r , the number of revolutions per minute, n , and the pull, P .

For, let p , be the pressure, and f , the coefficient of friction between the face of the drum D , and two brake blocks $W B$, then the twisting moment T , tending to turn the brake blocks round with the shaft is

$$T = 2 p f r_1$$

Where r_1 is the radius of the pulley or drum, D , in feet.

But this twisting moment is balanced by the pull on the spring balance, P , multiplied by its leverage, r .

$$\therefore 2 p f r_1 = P r.$$

The angle turned by the pulley or drum, D , per minute $= 2 \pi n$ radians, and since the work done by a couple, is the product of its moment into the angle through which the body turns :—

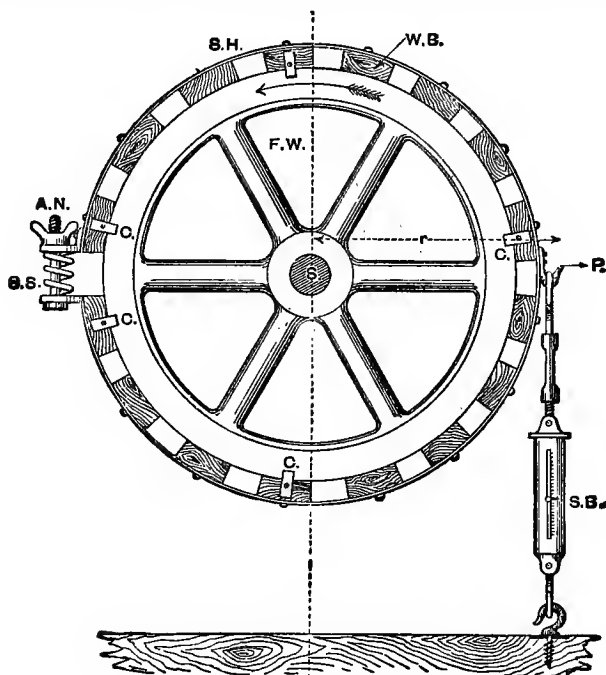
The work absorbed by friction = The work done per minute in foot-pounds.

$$\text{Or, } 2 p f r_1 \times 2 \pi n = P r \times 2 \pi n$$

$$\therefore \text{B.H.P.} = \frac{P r \times 2 \pi n}{33,000} = \frac{2 \pi r n P}{33,000}.$$

Improved Prony Brake. — Another very useful and practical form of Prony brake is that shown in the following figure. It is more suitable for larger powers and larger pulleys or flywheels than that shown by the previous illustration.

In this form of Prony brake the counterbalance weight, iron bar, and two wooden blocks, &c., are replaced by one or two thin steel straps, S H, fitted with a large number of hard wood blocks, W B, placed about 2 inches apart. These blocks are generally made of the same width as the flywheel, F W, upon which they bear, and they are kept from slipping to one side or the other by a number of metal clips, C, C, screwed on each



IMPROVED PRONY BRAKE DYNAMOMETER.

side of them. On starting the engine, the adjusting nut, A N, is left quite slack until the desired speed has been attained. It is then gradually turned until the necessary pull is registered by the Salter's balance, S B. Should this tightening up of the brake-strap raise the pointer, P, above the level line, P S, then the adjusting link between S B and P will have to be turned in a direction that will bring P down a little, when a slight slackening of A N will probably let P down to the level mark. The spiral

spring, S S, between the outstanding lugs of the brake-strap, serves to give the brake a little more elasticity than it would otherwise have, and also keeps these lugs hard against the head and nut of the adjusting bolt. After the desired speed and pull have thus been rendered fairly constant, a set of readings should be taken every ten or fifteen minutes over a period of several hours, and the mean B.H.P. obtained from the mean speed, n , and pull, P , and horizontal distance, r , in exactly the same way as in the previous example, viz.:—

$$\text{B.H.P.} = \frac{2 \pi r n P}{33,000}.$$

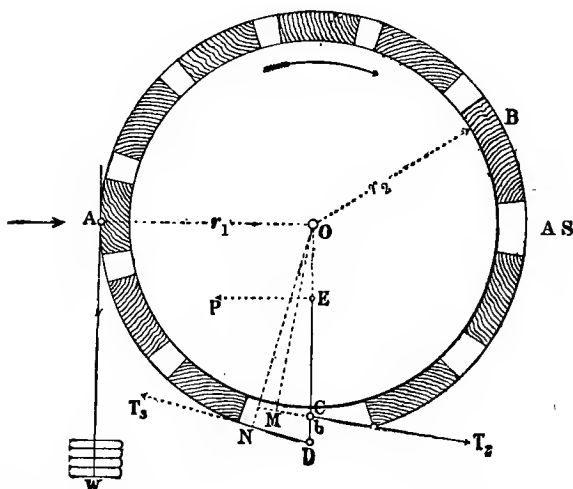
It will be evident from the figure, and from what we said about the ordinary Prony brake, that the Salter's balance may be replaced by a Pooley weighing machine resting on the ground, and a stiff vertical bar fixed between the bottom of the adjusting bolt and the platform of the weighing machine.

Appold's Compensating Lever.*—One of the best known forms of friction-brake dynamometers fitted with a compensating device, is that designed by Mr. C. E. Amos and Mr. Appold, and was the form used at one time for large powers by the Royal Agricultural Society. It is similar to that shown by the previous figure; but, besides a hand-adjusting screw at A S, it is provided with a compensating lever, E C D, by means of which the rise or fall of the load, W, is attended with a decrease, or increase, in tension on the brake-strap, so that a position of equilibrium may be automatically attained without causing inaccuracy in the indications. With a given tension in the brake-strap, and with the load, W, carried so that its point of suspension, A, is opposite the pointer mark, \longrightarrow , the lever, E C D, takes a vertical position; but as soon as the load, W, is lifted, the lever pivoted at E, moves round to the left hand, and virtually increases the length of the brake-strap, and thus slackens it, allowing the load again to descend. If, on the other hand, the total friction decreases and is insufficient to carry the load in its normal position, the descending load presses round the point of the compensating lever to the right, thus tightening the strap and increasing the frictional grip until the conditions are again such as will enable the lever to reassume the

* The following four figures are from *The Proc. Inst. C.E.*, vol. xcv., Session 1888-89, by kind permission of the Council, from a paper by W. W. Beaumont, M.Inst.C.E.; on "Friction Brake Dynamometers," which the student should consult, not only for the information contained in the paper, but also for that derivable from the excellent and extensive discussion.

vertical position. If the change in the position of the point of suspension of the load has been due to a temporary cause, this automatic action may restore the balance without further adjustment; but if the departure from the normal position is not small, then adjustment by hand-screw at A S must be resorted to. It will be seen that the compensating action cannot come into play except by the rise or fall of the weight from its proper position, and hence the value of the device is confined to its power of limiting that rise and fall.

So long as the Appold brake is not used for more than 15 H.P., and is sufficiently, but still sparingly, lubricated with tallow or suet, the friction between the wooden blocks and iron wheel is



APPOLD'S COMPENSATING BRAKE, AS USED BY THE ROYAL AGRICULTURAL SOCIETY.

sufficient at ordinary speeds to balance the load without tightening the belt-strap. Under these conditions the compensating lever does not sensibly affect the results, since the pull on it will not be more than a few pounds. The conditions are the same as, or very similar to, those which would obtain if the brake were without a compensating lever, but with a strap so slack that the bottom blocks barely touch the wheel.

In the correspondence upon Mr. Beaumont's paper, Professor

T. Alexander and Mr. A. W. Thomson considered that the Appold brake gave accurate results when it was used properly.

- Let W = Load on brake-strap (see foregoing figure).
 „ $T_2 T_3$ = Tensions at two ends, C and D, of strap connected to lower ends of compensating lever.
 „ P = Pull on upper end, at E, of this lever.
 „ $r_1 r_2$ = Radii of brake-strap and wheel respectively
 „ F = Total friction of brake-strap.

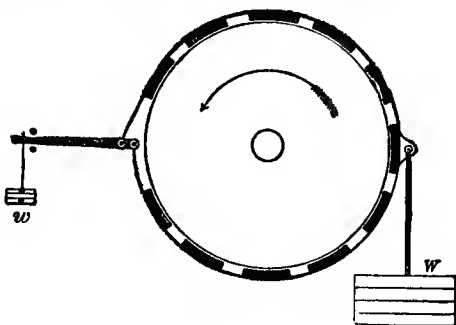
Suppose the lever, E C D, to take some definite fixed position, say to the left of the vertical when the engine is working smoothly. In this position, the lever may be supposed to be fixed to the ground. The tension of the brake-blocks on the lever towards the right at O, and left at D, are represented in the figure by T_2 and T_3 . On the other hand, the reactions of the lever on the brake-blocks are T_2 towards the left at C, and T_3 towards the right at D. Then, since there is equilibrium, the sum of the moments round the centre, O, of the weight, W , the friction of brake-blocks, and the tensions, T_2 and T_3 , is zero. Now, if we consider the lever as not fixed to the ground, but pivoted at E, then R, the resultant of T_2 and T_3 , must pass through E. T_2 and T_3 may now be replaced by R, and the sum of all the moments round O is again zero. Resolve R into vertical and horizontal components, V and P, acting at the point E. Since E is vertically under O, the line of action of V passes through O, and its moment is zero; and, therefore, the sum of the moments round the centre, O, of the weight, W , the friction of brake-blocks, and the horizontal force, P, acting towards the left at E, is zero; that is:—

$$W r_1 = F r_2 + P \times O E.$$

The amount of this horizontal force, P, can be easily measured by a spring-balance. With a low coefficient of friction, the tension on the brake-strap has to be increased; and since the ratio existing between T_2 and T_3 is constant, depending on the proportions of the lever, it follows that P may be of considerable amount; and any quantitative results calculated without taking it into account will be erroneous. With a high coefficient of friction the force, P, may be small, and the results might probably be not far wrong, even if P were left out of account. In every case, however, where accuracy is desired, the moment of P must be considered.

Professor A. B. W. Kennedy in his paper on the “Use and

Equipment of Engineering Laboratories"* says, that if the Appold pendulum-lever is used in any form for the automatic adjustment of the brake, it should be so arranged that its own pressure can be measured and allowed for. He prefers to use this brake in the manner illustrated by the accompanying figure where the small weight, w , is adjusted from time to time in order to keep the brake always floating freely. The changes in this weight have to be noted, and the necessary allowance made in the calculation for the B.H.P. He also believes that the side pressure, P , of the upper end of the pendulum-lever, as it was arranged in the Royal Agricultural Society brakes, if not measured and allowed for, causes a very considerable error in the calculated power. He also thinks that the brake should be large enough to run dry, as it is much more easily kept under control under these circumstances.†



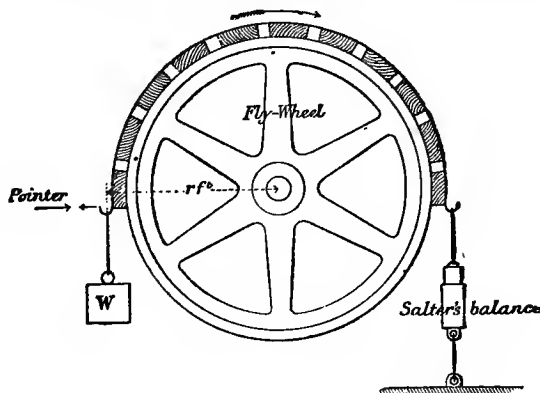
IMPROVED METHOD OF USING THE APPOLD BRAKE AS A DYNAMOMETER.

Semicircular Strap Dynamometer.—One very simple form of dynamometer which avoids the objections previously mentioned in regard to the Appold compensating lever, is shown by the following figure. Here, a semicircular strap of leather, or a number of $\frac{1}{8}$ - to $\frac{1}{4}$ -inch wires or steel bands, lined with hard wood, are attached at one end to a constant weight, W , lbs., and at the other end to a Salter's balance. The weight, W , should be tethered to some fixed bolt in the floor by a slack piece of flexible

* See vol. lxxxviii. (21st Dec., 1886) of *The Proceedings of the Institution of Civil Engineers*, London, for Prof. Kennedy's paper. Also see *The Mechanics of Machinery*, by Prof. Kennedy, published by Macmillan & Co.

† The student will observe that this method of using the Appold brake as a dynamometer, is the same as that referred to at the beginning of this Lecture, for restraining a submarine cable from passing too rapidly to sea.

rope in order to prevent the possibility of it being carried bodily over to the injury of the attendants in the case of the pulley firing and gripping the brake. If the flywheel or pulley revolve in the direction shown by the arrow, and the pointers, $\rightarrow \leftarrow$, are kept level with each other, then the net pull on the brake will be $(W - S)$ lbs., where S is the stress registered by the Salter's balance. Hence, if r be the horizontal distance in feet from the centre of the shaft to the vertical centre line of the weight, W ,



SEMICIRCULAR STRAP-BRAKE DYNAMOMETER.

as well as to the centre line of the Salter's balance, and n the number of revolutions per minute. Then :—

$$\left. \begin{array}{l} \text{The work done per minute on} \\ \text{the pulley and dissipated in} \\ \text{heat} \end{array} \right\} = 2 \pi r n (W - S) \text{ foot-lbs.}$$

$$\text{And, the B.H.P.} = 2 \pi r n (W - S) \div 33,000.$$

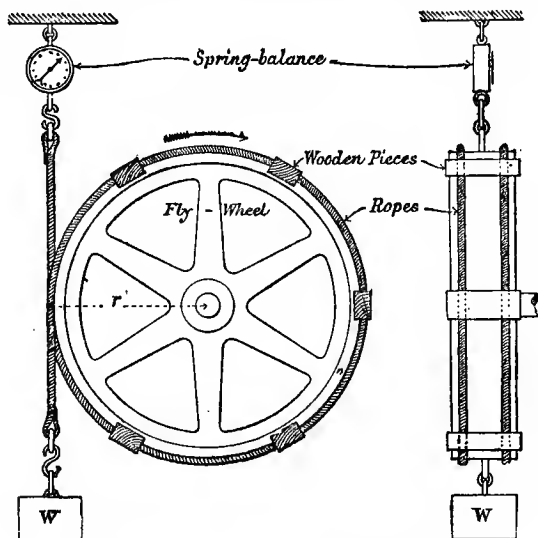
This form of absorption dynamometer has several objections :—

1. The lubrication requires considerable attention. (This fault is also common to all the previously mentioned dynamometers.)
2. The oil, grease, or soapy water used for lubricating the face of the brake-wheel bespatters the floor, &c., and the observer's clothes, unless the precaution is taken to thoroughly encase the lower half of the wheel. (This objection is also common to the previously mentioned dynamometers.)
3. If everything is not perfectly adjusted and running quite smooth, oscillations producing a hunting up and down action set in, due to variations in the friction; and consequently, considerable

guessing and frequent observations have to be taken of the Salter's balance.

These several objections are entirely obviated by adopting the rope-brake which we shall now illustrate and describe.



Society of Arts Rope Dynamometer.*—The jurors for the famous gas engine trials, held under the auspices of the London "Society of Arts" in 1888, were the first to publicly use a rope-brake in any extensive series of competitive trials, and hence the general name which has been given to this very simple and excellent form of brake. But, as will be seen from our footnote,



SOCIETY OF ARTS ROPE DYNAMOMETER.

rope-brakes had been designed and used prior to these tests by at least four well-known persons. As will be gathered from the

* The first rope-brake of which we have any record was invented by Sir Wm. Thomson, in 1872, and applied to his deep sea sounding machine as previously described in this Lecture. Prof. James Thomson, of Glasgow University, devised a rope-brake ergometer prior to 1880, see *Engineering*, Oct. 29, 1880, p. 379. An almost identical compensating rope-brake was also invented by M. Carpenter, of Paris, in 1880, see *Proc. Inst. C.E.*, vol. lxiii. (1881, part I.), p. 404. For a modification of this brake, by Prof. Barr, of Glasgow University, which is very similar to that afterwards used by the "Society of Arts," see *Proc. Inst. C.E.*, vol. lxxxviii. (1887, part II.), p. 110, and also vol. xc. (1889, part I.), p. 31.

following three sets of figures, this brake consists of an endless flexible rope, doubled round a pulley or the flywheel of an engine, and fitted with several  shaped wooden distance pieces, in order to keep the two parts of the rope uniformly apart and also to prevent them slipping off the wheel. These distance pieces or clips should be secured to the rope by soft copper wire lacing, drawn in from the outside of the clips and then through the centre of the rope, instead of being fastened thereto by nails or screws from the inside; for such latter metal fastenings are liable to part, to heat, and, consequently, char the rope. The rope should be thoroughly stretched and treated with castor oil or grease and black lead powder, prior to its being fitted to the wheel and to the clips; whenever long and important tests are desired. No further lubrication is required, and consequently the first and second defects mentioned on a previous page as pertaining to strap-brakes are entirely avoided. If large powers are to be demanded from a wheel of limited size, then it should have its rim of  section, so that a small stream of water may be played into the inside of the hollow part of the rim, which water will help very materially by its evaporation to dissipate the heat generated by the friction between the brake rope and the outer surface of the wheel. The surface of the pulley should be flat instead of rounded, in order to get the rope to work perfectly smooth, and a trial run of a few hours prior to the special test is advisable, in order to bring about a small flat glazed surface on the rope, which glazing is materially assisted by the previous application of the black lead powder. For anything up to 5 B.H.P. at 1,000 or more feet per minute of friction surface speed, the author has found that a flexible ship's log-line about .3 inch in diameter with a double turn round the wheel forms an excellent brake rope. From 5 to 10 B.H.P. a .5-inch diameter manilla rope serves the purpose. From 10 to 30 B.H.P. a .6-inch rope will do, and for 100 to 150 B.H.P. (at about 4,000 feet per minute) four turns of 1-inch rope on a large 16 feet diameter flywheel runs quite cool, as may be seen from the next figures on absorption dynamometers in this Lecture.

Advantages of the Rope-Brake.—The author has tested a large number and variety of motors with the rope-brake, and he considers that it has the following advantages:—

1. It can be constructed on short notice, from materials always at hand, in a factory or workshop, and at little expense.
2. It is so self-adjusting that very accurate fitting is not required.
3. It can be put on and taken off the brake-wheel in a very short time.
4. Being comparatively light and of small bulk, it can be hung up on the wall of the testing room, or laid past in a cupboard for future use.

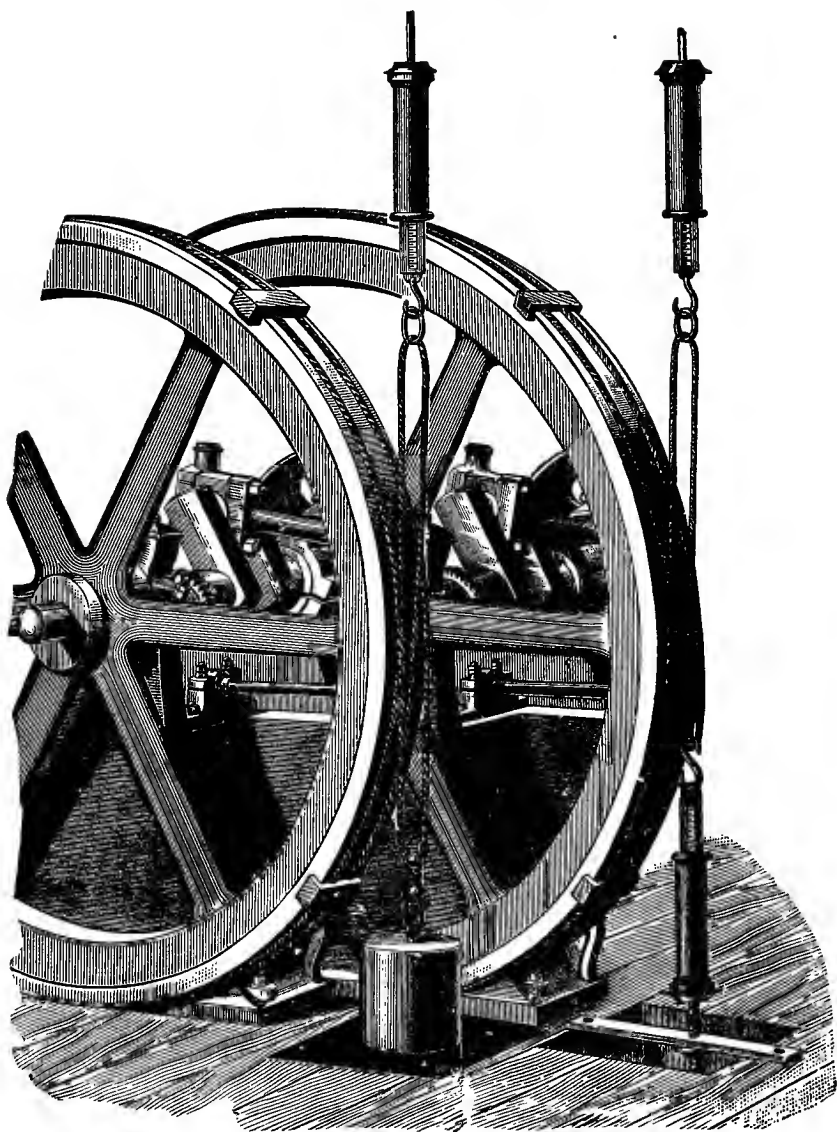


Fig. 1.

Fig. 2.

THE TWO FORMS OF ROPE-BRAKE

USED BY PROF. JAMIESON IN TESTING THE "ACME GAS ENGINE," AND
 "BROWN'S ROTARY ENGINE" FOR BRAKE HORSE-POWER.

5. It requires no attention whatever for lubrication, if the previously mentioned precautions as to treating and fitting the same are attended to.

6. The back pull registered by the spring-balance may be rendered very steady and of small amount by properly adjusting the weight, W , prior to the commencement of the recorded brake trials.

7. The brake-wheel, if of the proper size, soon attains a maximum temperature, so that the radiated heat equals that generated by the friction.

8. It may be used for very small as well as for large powers.

9. For large powers more and stronger ropes are only required on a comparatively larger wheel, and with the water cooling device mentioned in the previous section.

Tests of Engines with the Rope-Brake.—After what we have stated, three examples of such tests will suffice to show the student the wide variety of cases to which the rope-brake may be applied. The first is that of an Acme gas engine of about 19 B.H.P.; the second, that of a Brown's fast-speed rotary steam engine;* and the third, that of "Field's combined steam and hot-air engine."† The results of the first two are given in the first table, and those of the third in the second table, together with a graphic diagram of the more important conclusions.

Fig. 1 shows the arrangement of dead weight and Salter's balance used by the author in testing the "Acme gas engine," and Fig. 2 the way in which he applied two spring balances to the brake rope in case of "Brown's rotary engine." The latter plan has, under certain circumstances, particular advantages over the former. By the selection of two spring balances with different periods of oscillation, the tendency to jerk or "hunt" may be considerably reduced, or even entirely checked. Also, the nett brake load (i.e., P , the difference between the simultaneous indications on the two balances) may be kept constant throughout the test. This permits of the logarithm for $2\pi r P \div 33,000$ being ascertained and written down as a constant, prior to each observation, so that the only variable to be recorded is, n , the revolutions per minute. Consequently, the B.H.P. for each observation may be known within a minute or two after the mean speed has been noted, and the complete data may then be plotted to scale on a graphic diagram before taking the next observation.

* See *Proceedings of the Institution of Engineers and Shipbuilders in Scotland*, vol. xxxv., Session 1891-92.

† *Ibid.*, vol. xxxviii., Session 1894-95, for the author's papers on these tests.

B.H.P. TESTS OF AN "ACME GAS ENGINE," AND OF A
"BROWN'S ROTARY ENGINE."

DATA.	"Acme Gas Engine" by Alex. Burt & Co., Glasgow.	"Brown's Rotary Engine" by Lang & Sons, Johnstone.
Duration of tests in hours,	4	5
Initial gas or steam pressure in lbs. per sq. in. above atmosphere,	150	95
Final gas or steam pressure in lbs. per sq. in. above atmosphere,	1	1.5
Radius of brake load in feet,	2.771	2.042
Mean revolutions per minute,	154	574.5
Mean nett brake load in lbs.,	231	93.2
Mean B.H.P.,	18.77	20.8
Gas in cb. ft. or steam in lbs. per B.H.P.-hour, . .	19.13 cb. ft.	37.9 lbs.

The author was recently requested to test and report upon a new departure in the use of steam in steam engines. He has, therefore, much pleasure in placing the results of his experiments on "Field's combined steam and hot-air engine" before the student, because (1) they show one direction in which economy may be attained by preventing the condensation of steam in the cylinder; (2) the brake used was one of the largest in this country; (3) the table and the graphic diagram of results will form a useful example in case the student should be called upon to undertake similar tests.

This invention is the joint design of Mr. Edward Field, M.Inst.M.E. (inventor of Field's well-known tubular boiler), and Mr. F. Saunders Morris, M.Inst.M.E., working in conjunction with Messrs. Musgrave & Co., of Bolton, and Mr. George Dixon, their chief engineer.*

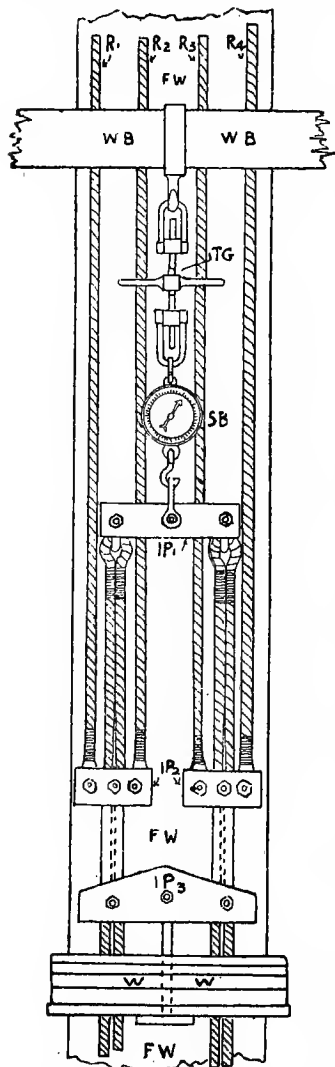
It consists of a hot-air pipe connection to the jacket and to each end of a single cylinder non-condensing engine.

A Roots' blower, driven by the engine, draws fresh cold air from the engine-room, and forces the same through a series of heating pipes placed in the main flue between the boiler or boilers and the chimney. This heater therefore occupies pretty much

* For a complete set of sectional figures of the cylinder, general arrangement of plant, and indicator diagrams, see the author's paper on this subject, vol. xxxviii., *Proceedings of the Institution of Engineers and Ship-builders in Scotland*, Session 1894-95, from which we have been kindly permitted by the Council to use the following two figures and extract of results.

the same position as a Green's economiser. The air was maintained, in my experiments, at a mean pressure of $1\frac{3}{4}$ lbs. on the square inch, and delivered to the ends of the cylinder at a mean temperature of 553° F., and to the valve-casing jacket at about 380° F. This hot air was admitted to the cylinder through special cylinder covers, each containing five inlet valves, which automatically opened inwards as soon as the exhaust steam commenced to escape. These valves continued open until compression commenced, being held close to their seats by light spiral springs. Consequently, the whole internal surface of the cylinder was heated up to a temperature far exceeding that of the steam, thus preventing the possibility of condensation taking place within the cylinder. Under these circumstances, the excellent result of 18.6 lbs. of steam per I.H.P.-hour was obtained from a single cylinder non-condensing engine—a result which, as far as the author can learn, has never been equalled by any other method of using steam in a single cylinder, and without subsequent condensation.

Brake Gear.—The large fly-wheel, of fully 50 feet in circumference and 20 inches in width, was used as a brake-wheel.



END VIEW OF BRAKE GEAR.

INDEX TO PARTS.

FW	represents flywheel.
R ₁ to R ₄	ropes.
WB	wooden beam.
TG	tightening gear.
SB	spring balance
IP ₁ , IP ₂ , IP ₃	iron plates.
W	weights.

TRIAL RUN OF FIELD'S PATENT (COMBINED STEAM AND HOT AIR) ENGINE.

[To face p. 180, Vol. I.]

Observations taken by Prof. JAMIESON at Messrs. Merryweathers, Greenwich, 30th November.

Permanent Data:—(1) Diameter of cylinder = 19.07 inches. (2) Diameter of piston-rod at front = 3.5 inches. (3) Effective piston area at back end = 285.6 square inches. (4) Effective piston area at front end = 276 square inches. (5) Mean effective area of piston = $A = 280.8$ square inches. (6) Length of stroke = $L = 2.997$ feet. (7) $\text{Log. } L \times A \div 33,000 = 2.4176$. (8) Brake weight radius = $r = 8.02$ feet. (9) Circumference of circle (whose radius = 8.02 feet) is = 50.3 feet. (10) $\text{Log. } 2 \pi r \div 33,000 = 3.1839$. (11) Number of boilers = 2. (12) Kind of boilers, Cornish. (13) Normal boiler pressure = 112 lbs. (14) Number of furnaces = 2. (15) Area of each fire grate = 10 square feet. (16) Sizes of feed-water tank (2 feet diameter \times 6 feet high) with long glass gauge tube and fixed scale marked off to gallons by careful weighing. (17) Weight of each fill of feed-water tank, from mark 10 to mark 110 on the fixed scale = 1,000 lbs. (18) Weight of each fill of the drain from steam pipe = 7 lbs.

Re INDICATED HORSE POWER.										Re BRAKE HORSE POWER.							Re HOT AIR SUPPLY.						Re FEED WATER SUPPLY.					
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	
Times	Mean Boiler Pressure by Gauge.	Initial Cylinder Pressure by Cards.*	Mean Effective Cylinder Pressure.			Harding's Counter Readings.	Strokes per Minuta.	I.H.P.	Steam per I.H.P. —hour in lbs.	Revs. per Minute.	Gross Brake Weight in lbs.	Back Pull in lbs.	Nett Brake Pull in lbs.	B.H.P.	Steam per B.H.P. hour.	Mech. Efficiency %.	Mean Pressure of Hot Air supplied (in lbs.)	Temp. of Air entering Blower.	Temp. of Air entering Jacket.	Temp. of Air leaving Jacket.	Temp. of Air entering Cylinder Front.	Temp. of Air entering Cylinder Back.	Temp. of Feed Water in deg. F.	Times of filling the Feed- Water Tank.	Number of fills of Feed- Water Tank.	Total Feed Water supplied (in lbs.)	REMARKS.	
			Back. in lbs.	Front. in lbs.	Mean. in lbs.																							
p.m.	lbs.																											
1.50	114	114 (C)	32.4	32.5	32.45	38684	162.0	137.5		81.0	961.5	112	849.5	105.1			2.0	...	395	380	460	490	60.5	p.m.		
2.10	114	113 (C)	33.0	33.4	33.2	40314	163.0	141.6		81.5	961.5	124	837.5	104.2			2.1	...	415	390	507	532	61	2.15	1	1,000		
2.30	114	114 (C)	33.0	33.2	33.1	41953	163.9	141.9		81.9	961.5	128	833.5	104.2			1.9	74° F.	426	398	525	555	59.5	2.36	2	2,000		
2.50	113	114 (C)	33.2	32.7	32.95	43591	163.8	141.2		81.9	961.5	128	833.5	104.2			2.0	...	410	370	531	570	59	2.58	3	3,000		
3.10	114	115 (W)	...	32.2	32.2	45231	164.0	138.1		82.0	961.5	126	835.5	104.6			1.75	...	405	360	540	585	59		
3.30	115	114 (W)	30.1	33.9	32.0	46870	163.9	137.2		81.9	961.5	124	837.5	104.7			1.75	...	385	345	535	595	59.5	3.24	4	4,000		
3.50	116	114 (W)	31.8	33.9	32.85	48503	163.3	140.4		81.6	961.5	122	839.5	104.6			1.7	78° F.	375	335	545	590	59.5	3.48	5	5,000		
4.10	115	114 (W)	31.7	32.7	32.2	50140	163.7	137.7		81.8	961.5	120	841.5	105.1			1.7	...	370	330	545	595	60.5	4.10	6	6,000		
4.30	113	114 (C)	32.8	31.6	32.2	51771	163.1	137.4		81.5	961.5	120	841.5	104.7			1.65	...	375	330	540	585	59	4.30	7	7,000		
4.50	114	114 (C)	30.5	32.0	31.25	53405	163.4	133.6		81.7	961.5	120	841.5	104.9			1.6	...	365	320	535	575	60	4.50	8	8,000		
5.10	113	113 (C)	31.5	32.0	31.75	55043	163.8	135.6		81.9	961.5	120	841.5	105.2			1.6	...	360	315	536	577	59.5	5.22	9	9,000		
5.30	111	111 (C)	30.5	30.5	30.5	56671	162.8	129.0		81.4	961.5	120	841.5	104.5			1.6	80° F.	358	315	539	578	59	5.37	10	10,000		
5.50	114	114 (C)	31.0	31.6	31.3	58301	163.0	133.5		81.5	961.5	122	839.5	104.5			1.6	...	372	365†	551	595	58		
6.10	113	113 (C)	30.7	31.4	31.05	59939	163.8	133.0		81.9	961.5	118	843.5	105.4			1.6	...	370	330	545	588	58	6.9	11	11,000		
6.30	112	112 (C)	30.4	32.4	31.4	61564	162.5	133.5		81.3	961.5	125	836.5	103.8			1.6	81° F.	365	320	543	586	58.5	6.34	12	12,000		
6.50	112	112 (C)	31.4	32.5	31.95	63201	163.7	136.8		81.8	961.5	120	841.5	105.1			1.6	...	360	320	550	595	58	6.50	13	12,780		
Means,	113.5	113.4	31.6	32.43	32.02	Total 24517	163.4	136.75	18.6	81.66	961.5	121.81	839.68	104.6	24.26	76.5	1.74	...	381.6	345.2	532.9	574.4	59.27	...	Total 13	12,689		


The circumference of the centre line of the brake load rope was taken as = 50.3 ft. instead of 50.8 ft. (the size used on previous trials). This reduces the mechanical efficiency about 1%. By the end of the five hours' test 81 lbs. of water had been drained from the steam pipe, and 10 lbs. from the bottom blow-off cock of No. 1 boiler. Thus making 91 lbs. to be deducted from the 12,780 lbs., the total feed water used, or a nett consumption of 2,537.8 lbs. per hour. During the five hours' test 1,680 lbs. of coal were burned, thus making the consumption of coal per I.H.P. hour = $\frac{1,680}{5 \times 136.75} = 2.45$ lbs., and per B.H.P. hour = $\frac{1,680}{5 \times 104.6} = 3.21$ lbs.

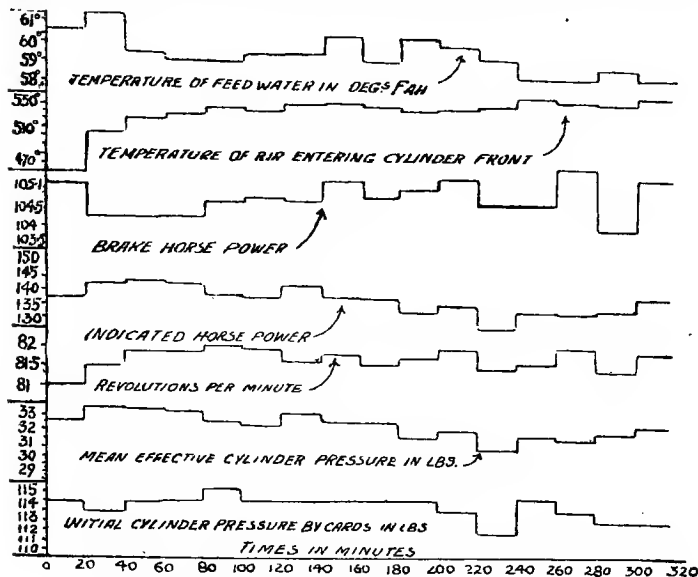
* C for Crosby Indicator.
W for Wayne Indicator. The spring used in this indicator was carefully compared with a duplex standard steam gauge made by Messrs. Schäffer & Budenberg.

† Rise of temperature here due to raising dampers.

The circumference of the centre line of the brake load rope was taken as = 50.3 ft. instead of 50.8 ft. (the size used on previous trials). This reduces the mechanical efficiency about 1%. By the end of the five hours' test 81 lbs. of water had been drained from the steam pipe, and 10 lbs. from the bottom blow-off cock of No. 1 boiler. Thus making 91 lbs. to be deducted from the 12,780 lbs., the total feed water used, or a nett consumption of 2,537.8 lbs. per hour. During the five hours' test 1,680 lbs. of coal were burned, thus making the consumption of coal per I.H.P. hour = $\frac{1,680}{5 \times 136.76} = 2.45$ lbs., and per B.H.P. hour = $\frac{1,680}{5 \times 104.6} = 3.21$ lbs.

This flywheel was encircled by four parts of a strong and flexible rope, 1 inch in diameter. The inner ends of this rope were attached to a spring-balance, tightening gear, and wooden beam, while the outer ends were connected to a fixed weight, consisting of nearly 1,000 lbs. of cast iron for the first day's trial, and about one-third of that for the second day's run.

In other words, the dynamometer was an excellent and large example of what has now come to be termed the "Society of Arts' brake." It worked perfectly, and there was no undue heating anywhere. This was no doubt partly due to the stream of water which played on the inside of the  shaped flywheel, to the large surface, and to the strong draught caused by the fan action



GRAPHIC DIAGRAM OF THE CHIEF RESULTS OF THE TESTS OF FIELD'S COMBINED STEAM AND HOT-AIR ENGINE.

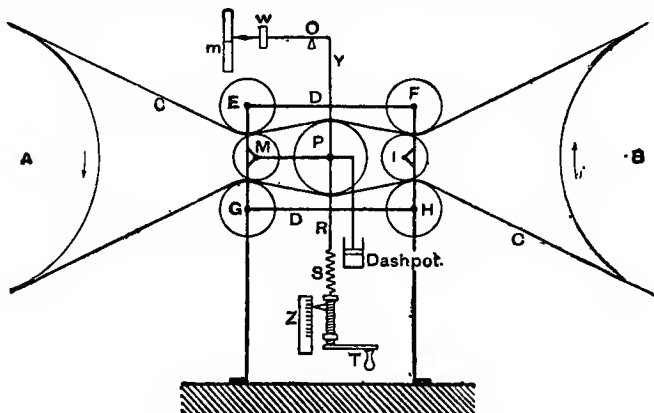
of the wheel. As far as possible, however, no water was permitted to get between the rope and the flywheel, and no lubrication of any kind was applied to these parts.

Counter.—The number of strokes and revolutions per minute were obtained by aid of a "Harding's counter" fixed to the crank shaft.

Tests.—First of all the permanent data marked at the top of the

table were carefully checked. Then, simultaneous observations were taken every twenty minutes of each of the items marked in columns 1, 2, 3, 7, 8, 11, 12, 13, 14, 18, 20, 21, 22, 23, and 24. The figures in columns 4, 5, and 6, relating to the mean pressure in the cylinder, were obtained from the indicator cards which were also taken from each end of the cylinders every twenty minutes. The figures in the other columns were either calculated or observed as required at the times stated in the respective columns. The more important observations and calculations are drawn to scale on the foregoing graphic diagram.

Transmission Dynamometers*—von Hefner-Alteneck or Siemens' Dynamometer.—Transmission dynamometers may be divided into two classes—(1) those which help to measure the work transmitted by a belt or set of ropes; (2) those which help to measure the work transmitted by a shaft. Of the first class, one form which has been used largely in dynamo tests is the Alteneck-Siemens' dynamometer. The general arrangement of this instrument is shown by the following diagram.† Power is transmitted from the



ALTENECK-SIEMENS' TRANSMISSION DYNAMOMETER.

* For a description of Morin's Traction, Rotatory, and Integrating Dynamometers, see Prof. Macquorn Rankine's *Manual of the Steam Engine and other Prime Movers*. For a description of Morin's, Webber's (similar to White's), Brigg's (modification of the Alteneck-Siemens'), Tatham's, Brackett's, Webb's, Hartig's, Emmerson's, Van Winkle's, and Flather's transmission dynamometers, see *Dynamometers and the Measurement of Power*, by John J. Flather, Ph.B., published by John Wiley & Sons, New York.

† The above figure is reproduced by permission from "The Electrician Series" of *Motive Power and Gearing*, by E. Tremlett Carter.

pulley A to the pulley B, through a flexible leather belt, C, which passes through the dynamometer. The apparatus consists of a rectangular framework, D, with idle pulleys pivoted at the four corners, E, F, G, H, and two other idle pulleys, I and M, also pivoted to the frame. In addition, there is a movable pulley, P, attached to a lever having its fulcrum at M, and its other end connected to a dash-pot or pump-brake, whereby any sudden jerking motion is lessened. The tight side of the belt tends to lift this pulley, P, while the slack side presses it down. Attached to the lever at the centre of P, are two vertical rods, R and Y. The rod, R, terminates at its lower end in a spiral spring, S, the tension of which is adjusted by the handle, T, and indicated on the scale, Z. The rod, Y, terminates at its upper end in a small lever, pivoted at O. This lever carries a weight, W, and is provided with a pointer which travels over a scale on which there is a zero mark, *m*. The instrument having been fixed in position, the handle, T, is turned so as to bring the pointer of the top lever to the mark, *m*. The reading on the scale, Z, is then noted, and the engine started. Instantly, the upper pointer will fall below the mark, *m*, on account of the pulley, P, being lifted; but this must now be rectified by turning the handle, T, so as to increase the pull of the spring, S, until the pointer again stands at *m*. The reading on the scale, Z, is again noted, and the former reading subtracted from it. Simultaneously with these observations, the velocity of the belt must be observed by measurement of the speed of the pulley B. Let *r* be the radius of the pulley B in feet, *n* its revolutions per minute, *P* the difference between the two readings on the spring scale, Z, and *k* a constant which is marked on the instrument. Then:—

$$\text{H.P.} = \frac{2 \pi r n P}{33,000 k},$$

which is the same formula we had before in the case of the Prony brake and the other forms of absorption dynamometers with the exception of the constant *k*.

The difference between the two readings on the scale, Z, constitutes the difference of the tensions in the rod R when the belt is doing work and idle, and is proportional to the difference of tensions in the two sides of the belt when driving the pulley B. The constant *k* is necessary from the fact, that only the vertical components of the tensions in the belt affect the rod R. If θ be the angle between the belt (where it leaves the pulley P) and the vertical diameter through P; T_d and T_s the tensions in the direct

or tight and slack sides of the belt respectively, and v the velocity of the belt in feet per minute (or $2\pi rn$), then :—

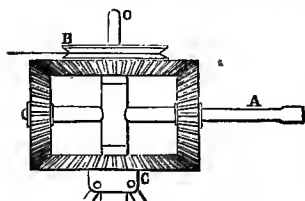
Nett pull on spring $S = P =$ Difference of tensions in rod R .

$$\therefore P = 2 T_d \cos \theta - 2 T_s \cos \theta = 2 \cos \theta (T_d - T_s).$$

$$\text{i.e., } T_d - T_s = \frac{P}{2 \cos \theta}, \text{ and } k = 2 \cos \theta.$$

$$\text{Again, the H.P.} = \frac{v(T_d - T_s)}{33,000} = \frac{2\pi rn \times P}{33,000 \times 2 \cos \theta} = \frac{2\pi rn P}{33,000k}.$$

Rotatory Transmission Dynamometers—Epicyclic Train,* or King's, White's, and Webber's Dynamometers.—The term “epicyclic train dynamometer” is applied to those of the second class of transmission dynamometers which help to measure the work in a shaft by transmitting the same through an epicyclic train. The effort exerted is measured by means of the force required to keep the train-arm at rest.



EPICYCLIC TRAIN DYNAMOMETER.

The following figure will serve to explain the principle (although not the full details) of King's, White's, and Webber's transmission dynamometers. The bevel wheel, B , is driven by a motor, and it transmits its motion through the intermediate wheels on the arm, A , to the bevel wheel, C , which is connected to the working machinery. The train-arm, A , is kept steady and level by a weight or spring attached thereto. There is usually a counter weight on a short extension of this arm, A , on the left-hand middle bevel wheel, which weight serves to balance the longer arm, A . Suppose the arm, A , to be permitted to revolve, then no work would be transmitted from B to C , and C would, therefore, remain stationary. In this case, the number of rotations of A , in a given time, would only be half that of B . Consequently, a weight placed on B at a certain radius from its centre would balance double that weight at the same radius on the arm, A . Therefore, the moment of the force applied to the arm, A (relatively to the common axis of A , B , and C), must be double the moment of the force transmitted from A to C when the arm, A , is balanced. Hence, if r be

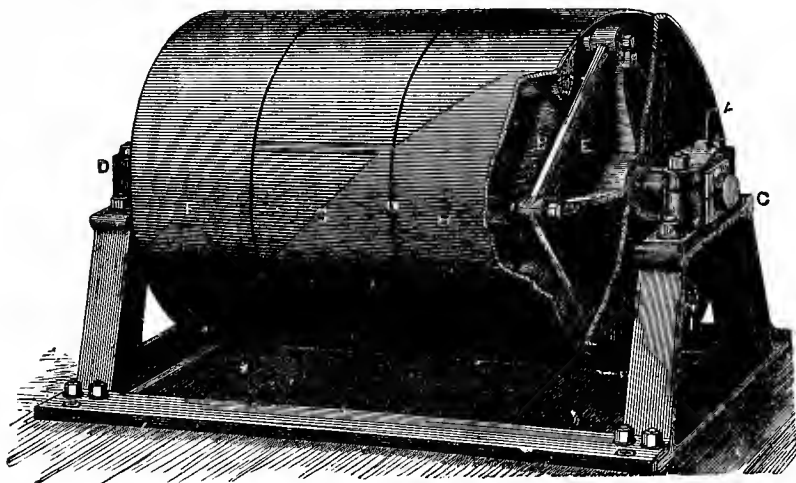
* The word epicyclic is derived from the Greek words *ἐπί*, signifying upon, and *κύκλος*, a circle. Hence this term for a wheel or wheels travelling around a circle or another wheel.

the radius of the pull, P , applied to A , and n the number of revolutions per minute of B and C , then :-

$$\left. \begin{array}{l} \text{The work transmitted} \\ \text{per minute} \end{array} \right\} = \frac{2 \pi r n P}{2}.$$

$$\text{And, the} \quad \text{H.P.} = \frac{2 \pi r n P}{2 \times 33,000} = \frac{\pi r n P}{33,000}.$$

Spring Dynamometers—Ayrton and Perry's and Van Winkle's **Transmission Dynamometers**.—Another kind of dynamometer belonging to the second class is that wherein springs, placed at



PROFS. AYRTON & PERRY'S TRANSMISSION DYNAMOMETER.

a certain radius from the centre of the rotating shaft, help to measure the torque therein when transmitting power.

One of the simplest and most easily understood is that devised by Profs. Ayrton and Perry, of the City and Guilds of London Technical Institute. A very similar instrument has been constructed by Mr. Van Winkle of U.S. America, and supplied to a firm in Chili for measuring up to 600 horse-power at 120 revolutions per minute. This one is believed to be the largest transmission dynamometer ever constructed.

The apparatus illustrated by the foregoing figure, consists of a pulley, F , rigidly fixed to the shaft, $C D$, a loose pulley, G , and

a pulley, H, joined by the spiral springs, B, to the ribbed plate, E, which is also rigidly fixed to the shaft, O D. If the motor belt be on F, and the belt to the dynamo or driven machine on H, or *vice versa*, the springs, B, will be stretched, depending on the "torque" or twist transmitted. The extension of these springs by means of a small link-motion (seen at the lower right-hand corner of the figure) causes the bright bead, A (at the end of a long arm), to approach towards the centre of the shaft. Hence, the smaller the radius of the circle described by this bright bead as it revolves, the greater the torque.* Consequently, the horse-power transmitted is at once obtained from observing the indicated torque and the speed of rotation. The arm carrying the bead is slightly flexible, and when no power is being transmitted the bead is pressed with a certain force against the rim of the front plate, hence the bead does not commence to move until a certain pre-arranged horse-power is being transmitted at a given speed. Its whole radial motion is, therefore, completed for a certain additional transmitted horse-power. The necessary addition depends on the strength of the springs and the leverage of the link-motion. Consequently, a large change in the radius of the circle of the bright bead is produced by a small change in the transmitted horse-power.

The next figure shows Profs. Ayrton and Perry's dynamometer coupling, which differs only from the preceding in that it is intended to be used with machinery driven directly by shafting where belting is not employed. For instance, this coupling may be used to measure the horse-power given by a fast-speed engine to a dynamo or other machine driven directly by it, or it may be employed to measure the power given by a marine engine to the screw or to the paddles, or generally the horse-power transmitted along any line of shafting; the spring coupling, in fact, replacing the ordinary coupling used with such shafts.

One of the halves of the coupling seen in the figure is keyed to the driving shaft—for example, the shaft of a fast-speed engine; and the other to the driven shaft—for example, that of the dynamo. The half, C, is attached to the other half by means of the spiral springs, and the stretching of these is therefore a measure of the torque. The angular motion of the one relatively to the other causes the bright bead, B, to approach the centre, and, as before, the radius of the circle of light helps one to measure the horse-power transmitted at any particular speed.

The transmission dynamometer and dynamometer coupling just

* The word torque was first suggested by Prof. James Thomson of Glasgow University, and means the turning moment or the turning force multiplied by its distance from centre of shaft.

described have the great advantage over any sort of laboratory dynamometers, in that the former have not to be put into position and adjusted for each particular experiment, but are always ready, and are always indicating the power transmitted at any given speed. If, for example, a dynamometer coupling be inserted in the shafting of a factory in place of the ordinary coupling, a glance at it, at any time, will show the power that is being transmitted by it. If two such dynamometer couplings be inserted at two places in the same set of shafting, the difference between the transmitted powers indicated by them is the power utilised by the machinery driven by that portion of the shafting that is between them.



PROFS. AYRTON & PERRY'S
DYNAMOMETER COUPLING.

Hydraulic Transmission Dynamometers — Flather's * and Cross's.—

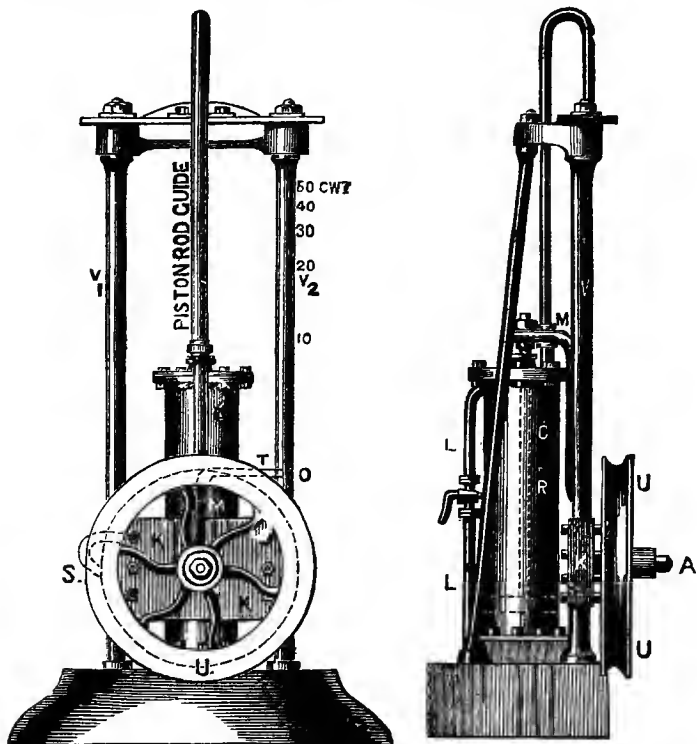
Owing to a want of confidence in results obtained by aid of spring dynamometers, Prof. Flather and Mr. J. A. Cross (both of the U.S. America) have independently perfected two forms of hydraulic transmission dynamometers, which are said to be reliable. The construction and action of these instruments are as follows:—The power shaft is keyed to a boss or pulley with two or more arms carrying hydraulic cylinders. The projecting ends of the plungers of these cylinders, bear upon the arms of a loose pulley on the same shaft. The torque imparted by the driving belt to the loose pulley is thus transmitted to the shaft through the liquid in the cylinders. The pressure thus caused in the liquid is conveyed by radial pipes to a common central trunnion, and from thence to a pressure gauge or indicator.

This apparatus has many advantages—(1) It is simple. (2) It is not affected to any great extent by the velocity of the shafting. (3) It requires no counter shaft, and no change of driving belt. (4) It takes the place of an ordinary driving pulley, and is driven by the same belt. (5) It may be connected to a recording gauge, and thus a continuous diagram of the load may be obtained with-

* Professor Flather's arrangement is described in his book on *Dynamometers and the Measurement of Power*, already referred to by two footnotes, and Mr. Cross's apparatus in *The Electrical Engineer* of New York, July 4, 1894, p. 3. An earlier form by von Hefner Alteneck is described in *Industries* of February 3, 1888, at page 122.

out any special attention. (6) It does not require to be displaced after a test is completed, for, by the simple closing of a cock or valve, the recording apparatus may be disconnected and the remainder left as an ordinary pulley.

Tension Dynamometer for Submarine Cables.—By referring to the figure of a telegraph steamer in a previous article of this



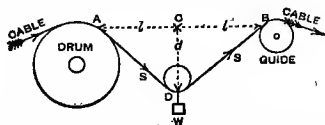
TENSION DYNAMOMETER FOR SUBMARINE CABLES.

Lecture, the student will understand the positions and use of the dynamometer on board a cable ship. As will be seen from the following figure, this apparatus consists of a vertical cylinder, C, filled with oil or soapy water, and containing a piston and a piston-rod, R, passing through a gland and stuffing box at M. This piston-rod is connected to a crosshead, K, which is free to move

up and down between the upright guides, $V_1 V_2$. On the outstanding turned pin, A, of the crosshead, K, there is carried an unkeyed grooved pulley, U, under which the cable is passed. To the top of the crosshead is fixed a pointer, T, which indicates the height to which the pulley, U, may be elevated, and consequently the stress on the cable or grapnel rope.

In order to prevent sudden jerks and oscillations of the moving parts, the top and bottom of the cylinder, O, are connected by a pipe, L, with a cock at its centre. The cylinder piston and liquid thereby act as a "pump-brake" or dash-pot, with greater or less freedom according to the opening of this cock. When paying out a heavy cable, or one in deep water, additional weights may be attached to the arm, A, in order to render the dynamometer less sensitive. In order to keep the pulley, U, always clean a curved scraper, S, is applied to the groove when desirable. The vertical scale may be marked off by calculation according to the following formula, but it should also be verified by an actual test, since this rule does not take friction into account.

In order that the friction between the dynamometer-crosshead, and its guides, shall be a minimum, the dynamometer (D in the following figure) *must* be placed midway between the point, A, where the cable bears on the paying-out or picking-up drum, and the point, B, where it bears on the guide-pulley next to the stern or bow sheaves, and these bearing points, A and B, should be in a horizontal line.



STRESS DIAGRAM FOR A SUBMARINE CABLE DYNAMOMETER.

- Let S = Stress on cable or rope (in cwts.) to be found.
 „ W = Weight (in cwts.) of all moving parts in dynamometer.
 „ l = Distance A C or C B (in inches).
 „ d = Deflection of cable from horizontal (in inches).

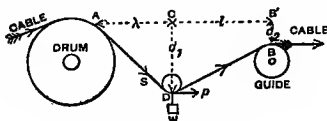
Then by the parallelogram of forces :—

$$S = W \frac{\sqrt{l^2 + d^2}}{2d} \text{ (cwts.)}$$

$$\therefore d = \frac{W l}{\sqrt{4 S^2 - W^2}} \text{ (inches).}$$

Since the stresses and deflections of the cable are approximately in inverse ratio to one another, and W and l are constant, it is only necessary to work out one example for d , plot it off on the dynamometer scale, and mark the others in the inverse ratio—*e.g.*, for double the stress half the deflection, and so on.*

If the points, A and B , are not in a horizontal line, and the dynamometer, D , not midway between them, as shown in the following figure, then the calculation becomes more complicated.



SPECIAL STRESS DIAGRAM FOR A SUBMARINE CABLE DYNAMOMETER.

Let S and W = Same as before (in cwts.)

„ p = Horizontal pressure on guides of D (in cwts.)

„ λ = Horizontal distance $A C$.

„ l = „ „ $C B'$.

„ d_1 = Deflection $C D$ (in inches).

„ d_2 = Vertical height $B B'$ (in inches).

$$\text{Then, } S = \frac{W}{\frac{d_1}{\sqrt{d_1^2 + \lambda^2}} + \frac{d_1 - d_2}{\sqrt{(d_1 - d_2)^2 + l^2}}}$$

$$\text{And, } p = S \left(\frac{l}{\sqrt{(d_1 - d_2)^2 + l^2}} - \frac{\lambda}{\sqrt{d_1^2 + \lambda^2}} \right)$$

If, however, the points, A and B , are in horizontal lines—*i.e.*, $d_1 = d$, and $d_2 = 0$, but D not midway between them.

$$\text{Then, } S = \frac{W}{\frac{d}{\sqrt{d^2 + \lambda^2}} + \frac{d}{\sqrt{d^2 + l^2}}}$$

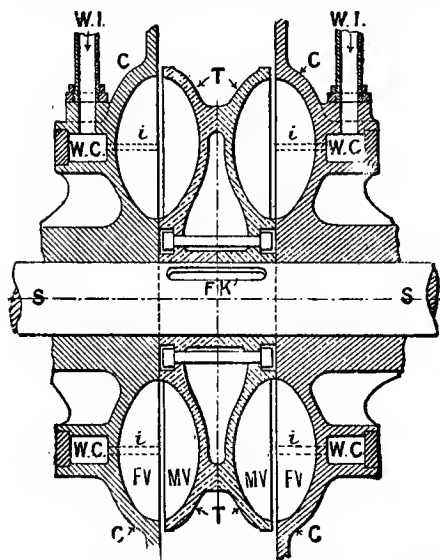
* The distance, l , between the guide pulley and the dynamometer is so great compared with the deflection, d , of the dynamometer, that the above rule is practically correct.

Froude's Water Dynamometer—General Description.—This machine was originally designed by the late Mr. William Froude for use in his famous experiments upon the propulsion of ships. It has since been modified for testing the brake horse-power of steam engines and other prime movers. These hydraulic brakes took the form of ordinary centrifugal pumps, wherein the resistance to rotation was varied to a certain extent, upon limiting the quantity of water in the brake by means of a valve placed in the outlet pipe. But to measure great powers, this form of apparatus would have had to be very large and cumbersome. To meet these difficulties Mr. Froude devised a new form of rotator and case, which gave a means of greatly increasing the resistance to rotation, as well as of regulating this resistance through a considerable range.* This dynamometer is generally coupled direct to the shaft of the engine or motor whose power is to be measured, but it can also be arranged for belt or rope driving.

The present up-to-date general arrangement of this kind of water dynamometer, as made by Messrs. Heenan & Froude, Ltd., of Worcester, for testing prime motors, is shown by the Frontis-plate and figs. 1 to 5. It consists of a casing or shell, C, supported by six friction rollers, R, which are provided with set-screws, so that the brake shaft, S, may be readily brought into line with that of the prime mover. There are three such rollers on each side of the casing, one pair being underneath and one pair on each side of the shaft. The rotator or turbine portion of the brake, T, is keyed to the shaft, S, as shown by the following vertical cross section. This rotator is formed of two sets of semi-elliptical cups set back to back, and turned by the shaft, S, between two outside similarly formed sets of fixed cups on the inside of the casing, C. These fixed and moving cups are provided with oblique vanes, F V and M V respectively, as

* See *Proceedings of the Institution of Civil Engineers*, vol. xcix., Part I., 1889-90, for Prof. Osborne Reynold's paper on "Tests of the Triple-Expansion Engines by this Brake," and for his modifications. Also his application of this brake for re-determining Joule's Mechanical Equivalent of Heat, as explained in his Bakerian Lecture before the Royal Society of London, May 20, 1897, and reproduced in his *Scientific Papers*, vol. ii., Cambridge University Press, 1901. He therein proves by extensive accurate experiments, that the mean of his results gave 777 ft.-lbs., instead of Joule's equivalent of 772, for 1 British thermal unit. Prof. Reynold's modification, was to work with the dynamometer only partly filled with water, and the variation of the amount of water contained by the brake varied the power absorbed; whereas, Messrs. Heenan & Froude keep the brake always full of water, and vary the freedom of circulation or power absorbed, by sliding forward a thin metallic shield or flat plate between the faces of the fixed and moving vanes.

shown by the horizontal cross section. It therefore follows, that if water be admitted to these cups it will be violently churned and circulated between the rotating and fixed sets of vanes; thus offering a natural resistance to the turning of the central turbine, and converting the power of the prime motor directly into heat by raising the temperature of the water as it circulates to and from or amongst these channels.



INDEX TO FIGS. 1 to 5.

- W I for Water inlets.
- W C „ Water channels.
- i i „ Inlet holes.
- F V „ Fixed vanes.
- M V „ Moving vanes.
- T „ Turbine.
- S „ Shaft.
- F K „ Feather key.
- C „ Casing.
- G „ Glands (figs. 3 to 5).
- H „ Handwheels „ „
- P „ Plate sluices „ „
- R L „ Radius lever „ „

FIG. 1.—VERTICAL SECTION OF FROUDE'S DYNAMOMETER,
AS SHOWN BY FRONTISPLATE, AND MADE BY HEENAN & FROUDE.

Details of the Dynamometer.—Cold water under a pressure of 15 to 20 lbs. per square inch is admitted to the brake by the supply pipe, S P, and its valve, to the centre of the branch pipes or two water inlets, W I, where its pressure is measured by the gauge as shown upon the general view. By comparing the vertical and horizontal sections it will be seen, that the water then passes into circular water channels, W C, on each side of the main casing of the brake. From these two channels it is led through small holes or inlets, i i, in the fixed vanes, F V, to the cups formed by the moving vanes, M V, of the rotator or turbine, T. It then leaks under the action of centrifugal force through the outside narrow space between the faces of the fixed

and moving vanes, and enters the annular chamber between the outer periphery of the turbine wheel and the cylindrical portion of the casing.

By keeping the afore-mentioned hydrostatic pressure at 15 to 20 lbs. on the supply water as it enters the centres of the moving cups, and by placing the water outlet pipe, W O, at the top of the brake, air is prevented from accumulating at the centres of the whirlpools set up by the violent vortex motion in each of the cups. We thus have a continuous supply of cold water entering the brake by the W I inlet pipes, and leaving it at a higher temperature by the W O outlet pipe. The heat units imparted to the water by the conversion of mechanical to heat energy may be estimated by multiplying this rise in temperature by the weight of water which flows through the brake in unit time.

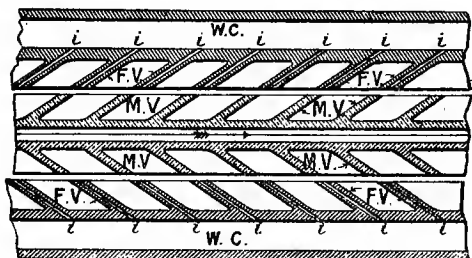
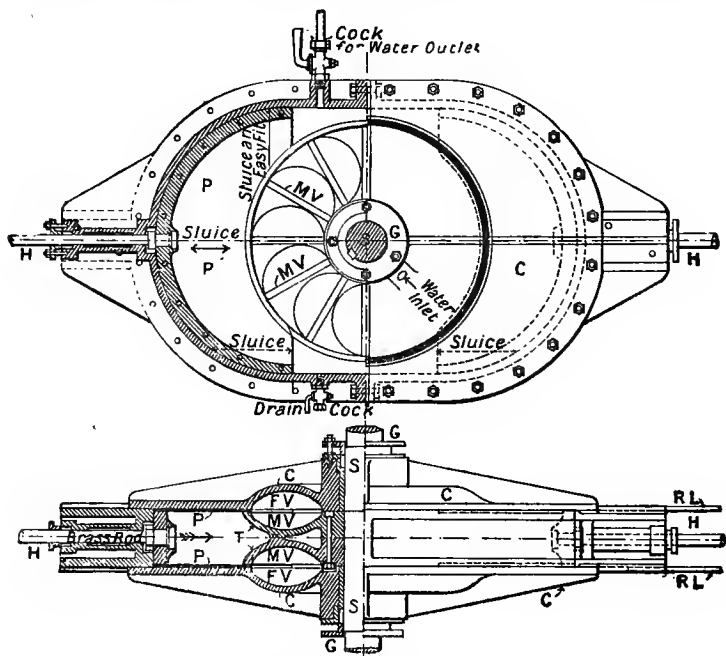


FIG. 2.—HORIZONTAL SECTION THROUGH WATER CHANNELS, FIXED AND MOVABLE VANES OF FROUDE'S DYNAMOMETER.

Froude's Dynamometer as used by Prof. Weighton at Durham College of Science.—A cock on the water outlet pipe, W O, regulates the discharge, so that it does not attain too high a temperature. Prof. Weighton found, that the best working temperature for the discharge water was between 120° and 150° F. The water is prevented from passing out between the casing and the main shaft by the use of stuffing-boxes, packing, and glands. But, it will easily be seen, that no error is made by their friction, as this reacts on the casing in the same direction as the water. An oil-pump, O P, and oil pipes are provided for supplying a lubricant to the glands and bearings. It will be observed that owing to the use of flexible water-pipe connections, the dynamometer as a whole can be readily lifted by the two links, L L, shown on the general outside view. Prof. Weighton, in his tests of the Quadruple Expansion Engines at the Durham College of Science, Newcastle-on-Tyne, used a brake of similar design to this one, and students are referred to the *Transactions of the North-East Coast Inst. of Engineers and Shipbuilders*, vol. xiii., session 1896-97, for his paper, with results of tests and discussion thereon. I am indebted to the Council of this Institution for their kind permission to reproduce the three figures

showing part of the water dynamometer illustrating the shafts, stuffing-boxes with glands, G, and the method of varying the range of the brake by sluices or shield plates, P, for absorbing different powers.

Action of the Dynamometer.—The action and reaction which is set up by the water between the central rotator or turbine, T, and the freely suspended outer casing, C, tends to turn the casing in the same direction as the rotated turbine. This torque is balanced by attaching a radius lever, R L, to the casing as shown by the Frontisplate, and connecting its outer end to



FIGS. 3 AND 4.—SHOWING FROUDE'S DYNAMOMETER AS USED BY PROF. WEIGHTON AT DURHAM COLLEGE OF SCIENCE.

known weights, w and W . Before starting the brake, it is necessary to have its balance adjusted by the "Denison Balance," B, so that everything may be in equilibrium before the weights are applied to counteract the torque of rotation. It is usual to make the length of the radius lever, R L, from the centre of the shaft, S, to the point of suspension of the

weights 5 feet 3 inches, so that the circumference of a circle of this radius may be exactly 33 feet. By doing so, the calculation for the B.H.P. is simplified, as shown by the ordinary formula for brake horse-power.

$$\text{Where,} \quad \text{B.H.P.} = \frac{2 \pi r n P}{33,000}.$$

Here, $2 \pi r = 33$ feet; $P =$ net pull, and $n =$ revs. per minute.

$$\text{Hence,} \quad \text{B.H.P.} = \frac{33 n P}{33,000} = \frac{n P}{1,000}.$$

This is a readily and easily manipulated value. In order to

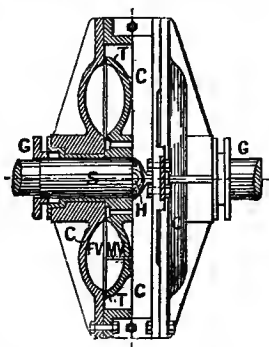


FIG. 5.—END VIEW AND SECTION OF FIGS. 3 AND 4.

secure steady readings, it is found convenient to provide a greater weight ($w + W$), than is actually required to balance the torque of rotation. Consequently, this excess of weight, which is not equalised by the torque, is adjusted by the counter-effect of the balance, B , when the radius lever is level. The net pull or effective weight acting at the 5 feet 3 inches radius, is the *total weight minus the balance reading*, S . Or, $P = (W - S)$, in exactly the same way as previously described for the strap or rope dynamometers.

Adjusting Water Brake to suit Various Brake Powers.—The power absorbable by this dynamometer is easily adjusted by turning the hand-wheels, $H H$, which move two thin metallic shields, sluices, or plates, $P P$, between the faces of the fixed and rotating vanes, thus cutting off more or less of the vortex action between the fixed and moving cups (see Figs. 3 and 4). In the various sizes of the brakes turned out by the makers,

the adjustment may be regulated from full power to one-fourteenth of this amount. This brake places in the hands of electrical and mechanical engineers a handy means of accurately measuring the power developed by their prime motors up to almost any conceivable size. For, with a rotator of only $15\frac{1}{2}$ inches diameter, 50 B.H.P. can be absorbed at 450 revs. per minute; and, with one of $30\frac{1}{2}$ inches diameter, 1,100 B.H.P. can be measured at 350 revs. per minute, down to 550 B.H.P. at 250 revs. per minute. If larger sizes are objected to, then two or more such dynamometers may be coupled in series.

LECTURE VIII.—QUESTIONS.

1. State the various ways in which friction may be applied usefully. Sketch and describe a good friction clutch or coupling. State the advantages and disadvantages of friction clutches.

2. Sketch, with an index to parts, and give a concise description of, the following pieces of mechanism—(1) Addyman's friction coupling; (2) Bagshaw's hollow sleeve clutch.

3. Sketch and describe (1) Weston's friction coupling and brake; (2) Weston's centrifugal friction pulley; (3) Robertson's grooved disc friction coupling.

4. Sketch and describe a friction brake as applied to a crane. The lever applied to the strap is a bent lever, of which one arm is 2 feet 11 inches long, and the other arm, which is at right angles to it, is $3\frac{3}{4}$ inches long, the diameter of the friction drum being 2 feet; find the tension of the movable end of the strap when a pressure of 100 lbs. is applied to the handle, and the tension at the fixed end for a given coefficient of friction. *Ans.* 933.3 lbs.; 1,495 lbs. or 582.6 lbs. according to direction of rotation, taking $\mu = 0.1$, and $\theta = 270^\circ$.

5. With the assistance of sketches describe the construction of two kinds of brakes, one in which a resisting force of moderate magnitude is overcome through a considerable distance, and the other in which a considerable resistance is overcome through a comparatively small distance. (*S. & A. Hons. Mach. Const. Exam., 1895.*)

6. Prove by a skeleton sketch and mathematical investigation the proper direction of rotation of a brake-wheel (with respect to its strap and lever connections) in the case of a winch or crane when the load is being lowered.

7. A strap, bearing on a brake wheel 2 feet in diameter, and tightened by a lever, is used to hold the load on a winch. The shaft to which the brake wheel is keyed also carries a pinion of 10 teeth, gearing with a wheel of 54 teeth on a second shaft. This second shaft has a pinion of 9 teeth gearing with another wheel of 50 teeth on the drum shaft. The diameter of the drum is 12 inches, the length of the handle of the lever is 30 inches, and of the short end 3 inches. If one end of the strap, which subtends an angle of 300° at the centre of the wheel, be fixed, and the other attached to the short end of the lever, find the greatest load on the rope wound on the drum that could be supported by a force of 45 lbs. applied at the end of the lever handle. Take $\mu = 0.1$. *Ans.* 18,576 lbs.

8. If a weight of 16,100 lbs., attached to the rope in the last question, is descending with a velocity of 300 feet per minute, find how far it will go after the brake is put on before coming to rest. The kinetic energy of the wheels may be neglected. *Ans.* 4 inches.

9. Explain the use, construction, and position of brake wheels in telegraph cable steamers.

10. Sketch and explain Lord Kelvin's deep-sea sounding machine, including a side elevation and plan of his differential rope-brake for the same.

11. Describe a method of obtaining the brake horse-power of an engine, and state the advantages to buyer and seller of adopting this method over that of nominal or indicated horse-power. An engine is making 150 revolutions per minute, the diameter of the brake pulley being 4 feet, and the pull on the brake 50 lbs., what is the B.H.P.? *Ans.* 2.85.

12. Sketch, and describe with an index to parts, some good form of absorption friction dynamometer. The pulley on the crank shaft to which

the brake is fitted is 3 feet in diameter, and makes 100 revolutions per minute. When the engine is at work, a Salter's balance, fixed at a point 21 inches from the axis of the shaft, registers 200 lbs. Find the brake horse-power of the engine. Prove the formula you use. *Ans.* 6.6 H.P.

13. Describe the ordinary friction dynamometer. If the shaft of an engine being tested makes 20 revolutions per minute, and the weight supported be 200 lbs., the point at which it is supported being 3 feet from the axis of the dynamometer, find the horse-power of the engine. *Ans.* 2.28 H.P.

14. In a friction brake dynamometer a weight of 93 lbs. is hung at a distance of $31\frac{1}{2}$ inches from the centre of the wheel. The brake wheel is driven by a pulley 5 feet in diameter, on the same axis, which carries a belt from the flywheel of an engine and makes 200 revolutions per minute. Explain the theory of the apparatus and find the horse-power exerted by the engine. *Ans.* 9.3 H.P.

15. State and prove wherein you consider "Appold's Compensating Lever Brake Dynamometer" defective. What precautions or alterations in this apparatus should be given effect to, in order to obtain accurate results with it?

16. Sketch and describe the rope-brake dynamometer, and state its advantages over other forms of absorption dynamometers for ascertaining the B.H.P. of an engine. What benefits are claimed in certain cases for the use of two spring balances instead of a weight and a spring balance with this apparatus? A flywheel is 10 feet diameter and rotates at 100 revolutions per minute, whilst the mean dead load is 1,000 lbs., and the back pull 100 lbs. Find the B.H.P. Supposing that the mechanical efficiency of the engine is 80 per cent., what would be the corresponding I.H.P.? *Ans.* 85.7 B.H.P.; 107 I.H.P.

17. Explain the epicyclic train form of transmission dynamometer, and prove the formula for the same.

18. Explain by a sketch and index to parts, a transmission power dynamometer by which the difference in the tensions of the two sides of the driving belt is measured. Explain the advantages of this instrument over the absorption dynamometer.

19. Explain and illustrate a form of spring transmission dynamometer and coupling.

20. Explain and illustrate a form of hydraulic transmission dynamometer.

21. Explain and illustrate a tension dynamometer as used in the paying out or picking up of submarine cables, and indicate by sketches where this apparatus is placed on board a submarine cable steamer. What are the most advantageous conditions for the employment of such a dynamometer? Prove the formula for graduating the scale.

22. Distinguish between an absorption and a transmission dynamometer. Describe the action and sketch the construction of an "epicyclic train form of dynamometer," and obtain an expression for calculating with the aid of such an apparatus the horse-power being transmitted by a shaft. The power of a portable engine is tested by passing a strap over the flywheel, which is 54 inches in diameter; one end of the strap is fixed, while a weight is suspended from the other end. With such an arrangement what would be the horse-power transmitted by the engine when running at 160 revolutions per minute, if the suspended weight is 300 lbs. and the tension on the fixed end is found by a spring balance to be 195 lbs.? *Ans.* 7.2 H.P.

23. You are asked to test the efficiency of a water motor or an electro-motor (make a choice). Describe, with sketches, how you would proceed.

24. Describe, with sketches, a brake suitable for about 10 horse-power from a gas engine. How would you calculate the brake-power? If, when the brake-power is 9·7, the indicated power is 11·8, and when the brake-power is 0 the indicated power is 1·75, what is the probable brake-power when the indicated power is 7·25? Use squared paper or not as you please. State the reasons for your method of arriving at the result.
Ans. 5·3 H.P.

25. You are asked to find the performances and efficiencies under various steady loads of a large gas engine using Dowson gas. It has two flywheels, and its maximum brake horse-power is about 100. Make good sketches of the brake or brakes; how would you keep it or them cool? You need not describe carefully how we take indicator diagrams; but you must state what precautions are to be taken for accuracy. What gas measurements would you take? How would you find the heat given to the water-jacket? What sort of results as to brake and indicated power would you expect to find?

26. Describe a Froude dynamometer for measuring the power given out by a steam engine of about 200 indicated horse-power. Find the brake power if 1,100 gallons of water are discharged per hour with a rise of temperature of 36 Fahrenheit degrees. *Ans.* 154·8 B.H.P.

(B. of E., H., Part I., 1900.)

N.B.—See *Appendices B and C* for other questions and answers.

LECTURE VIII.—A. M. INST. C. E. EXAM. QUESTIONS.

1. Sketch and describe a form of rope-brake suitable for measuring the work done by an engine of say 10 H.P. What quantities have to be observed in measuring the brake-power, and how is it calculated from them? (I.C.E., Oct., 1897.)

2. An electric motor is held in a cradle which is free to swing on knife edges in line with the axis of the motor. The motor drives a lathe from a pulley 6 inches in diameter by means of a belt running at 22 feet per second. It is found that a weight of 5 lbs. applied at a radius of 15 inches is required to keep the cradle from tilting sideways when the lathe is being driven. What horse-power is the motor exerting? (I.C.E., Feb., 1898.)

3. Describe a form of rope-brake to be used as an absorption dynamometer in measuring the power exerted by a motor of about 1 H.P., and show how to express the power in terms of the observed quantities. How may such a brake be made to adjust itself automatically to give a uniform moment of resistance? (I.C.E., Feb., 1899.)

4. Describe a form of transmission dynamometer, and explain how the power transmitted is measured by it. (I.C.E., Feb., 1900.)

5. Explain how to find the horse-power that an engine is developing. (I.C.E., Feb., 1900.)

6. Describe the construction and mode of use of a simple absorption dynamometer adapted to the measurement of the H.P. given out by an electric motor, and show how the H.P. is determined. (I.C.E., Oct., 1900.)

7. A wheel 12 feet in diameter, rotating at the rate of one revolution in 2 seconds, is acted on by a brake which applies normal pressures of 1 cwt. each at opposite ends of a diameter. If the coefficient of friction be 0.6, find (in horse-power) the rate at which work is being absorbed. (I.C.E., Feb., 1901.)

8. Explain how the horse-power of an engine may be determined by using a rope or band brake. Find the B.H.P. of an engine when the load on the brake is $1\frac{1}{2}$ cwt., and the pull at the free end 5 lbs., the flywheel being 5 feet diameter, the revolutions 120 per minute, and the thickness of the rope $\frac{3}{4}$ inch. (I.C.E., Oct., 1901.)

9. In what respect has the "tail-rope" brake, for measuring the power of an engine, an advantage over the "Prony"? In a trial with the tail-rope the spring balance shows a pull of 180 lbs. at 150 revolutions. The wheel is 5 feet in diameter and the weight 300 lbs. Find the horse-power. What would be the effect of a decrease of 10 per cent. in the friction, due to increased lubrication, the weight being unaltered? (I.C.E., Oct., 1902.)

NOTE.—The "tail-rope" brake here mentioned is the same as that illustrated in Lecture VIII., and termed the "Society of Arts Rope Dynamometer or Brake."

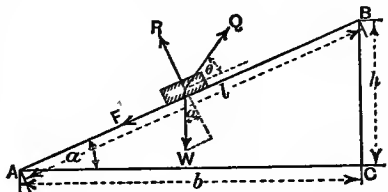
10. With the aid of sketches describe one form of dynamometer for measuring the turning moment exerted on a revolving shaft and describe the process of measurement. (I.C.E., Oct., 1904.)

N B.—See Appendices B and C for other questions and answers.

LECTURE IX.

CONTENTS.—Inclined Plane—Examples I., II., and III.—The Double Inclined Plane—Examples IV. and V.—Screws—Efficiency of Screws—Maximum Efficiency of Screws—Non-Reversibility of Ordinary Screws and Nuts—Tension in Bolts due to Screwing up—Example VI.—Speed Reducing Worm Gear—Questions.

Inclined Plane.—We now proceed to determine the relation between Q and W in the inclined plane when friction is taken into account. The most general case occurs, when the direction of Q makes a given angle θ , with the inclined plane AB . We shall, therefore, consider this case first, since all other particular cases can be easily deduced therefrom.



GENERAL CASE OF THE INCLINED PLANE.

Let R = Reaction perpendicular to the plane.

„ μ = Coefficient of friction.

„ $F = \mu R$ = Friction between body and plane.

By the “*Principle of Work*,” we get :—

Work done by Q = *Work done on W* + *Work done against F* .

Suppose the body to be dragged along the plane a distance $AB = l$.

Then :—

$$\text{Work done by } Q = Q \cos \theta \times AB = Q l \cos \theta.$$

$$\text{Work done on } W = W \times BC = Wh.$$

$$\text{Work done against } F = F \times AB = \mu R l.$$

$$\therefore Q l \cos \theta = Wh + \mu R l.$$

We must now eliminate R . The simplest way to effect this, is to consider the equilibrium of the forces acting on the body when Q is just about to draw the body up the plane.

Resolving the forces at right angles to AB , we get :—

$$R + Q \sin \theta = W \cos \alpha.$$

$$\therefore R = W \cos \alpha - Q \sin \theta.$$

By substituting this value of R in the above equation, we get:—

$$Q l \cos \theta = W h + \mu W l \cos \alpha - \mu Q l \sin \theta.$$

$$\therefore Q l (\cos \theta + \mu \sin \theta) = W (h + \mu l \cos \alpha).$$

Or,

$$\frac{Q}{W} = \frac{h + \mu l \cos \alpha}{l (\cos \theta + \mu \sin \theta)} \quad \dots \quad (i)$$

This equation can be put into a more convenient form, thus:—

$$\frac{Q}{W} = \frac{\frac{h}{l} + \mu \cos \alpha}{\cos \theta + \mu \sin \theta}.$$

But, $\frac{h}{l} = \sin \alpha$; and $\mu = \tan \phi = \frac{\sin \phi}{\cos \phi}$; where ϕ = angle of friction.

$$\therefore \frac{Q}{W} = \frac{\sin \alpha \cos \phi + \sin \phi \cos \alpha}{\cos \theta \cos \phi + \sin \theta \sin \phi}$$

i.e.,

$$\frac{Q}{W} = \frac{\sin (\alpha + \phi)}{\cos (\theta - \phi)} \quad \dots \quad (II)$$

From this *general* equation the results for any particular case can be deduced.

Case I.—*Suppose the plane to be smooth.* Then $\phi = 0$ and $Q = P$, the theoretical force required.

$$\therefore \frac{P}{W} = \frac{\sin \alpha}{\cos \theta} \quad \dots \quad (II_a)$$

Case II.—*Suppose Q acts parallel to AB , then $\theta = 0$.*

$$\therefore \left. \begin{aligned} \frac{Q}{W} &= \frac{\sin (\alpha + \phi)}{\cos \phi} \\ „ &= \sin \alpha + \mu \cos \alpha \\ „ &= \frac{h + \mu b}{l} \end{aligned} \right\} \quad \dots \quad (II_b)$$

$$\dots \quad (II_c)^*$$

$$\therefore Q l = W h + \mu \cdot W b \quad \dots \quad (II_d)^*$$

Or stated in words:—

The work done in raising a body up a rough inclined plane is equal to the work done in lifting it vertically through the height of the plane, together with the work done in dragging it along the base supposed to be of the same roughness as the plane itself.

* These are two important results which we shall frequently refer to in what follows.

In this case, when Q is parallel to AB , we get:—

$$\text{Actual Advantage} = \frac{W}{Q} = \frac{\cos \phi}{\sin (\alpha + \phi)} \quad \dots \quad (\text{III})$$

Or, from (II_c),

$$\text{Actual Advantage} = \frac{l}{h + \mu b} \quad \dots \quad (\text{IV})$$

$$\text{Efficiency} \dots = \frac{W h}{Q l} = \frac{W}{Q} \sin \alpha = \frac{\sin \alpha \cos \phi}{\sin (\alpha + \phi)} \quad (\text{V})$$

Or, from (II_c),

$$\text{Efficiency} \dots = \frac{h l}{h l + \mu b l} = \frac{h}{h + \mu b} \quad \dots \quad (\text{VI})$$

Case III.—Suppose Q acts parallel to AC . Then $\theta = -\alpha$.

$$\therefore \quad \frac{Q}{W} = \frac{\sin (\alpha + \phi)}{\cos \{-(\alpha + \phi)\}} \quad [\text{equation (II)}].$$

$$\text{i.e.,} \quad \frac{Q}{W} = \tan (\alpha + \phi) \quad \dots \quad (\text{II}_e)^*.$$

An application of this case will be met with when we come to treat of screws.

We shall now prove the following:—

PROPOSITION.—For a given inclination, α , of the plane and angle of friction, ϕ , the effort Q will have its least value when $\theta = \phi$; i.e., when the direction of Q makes an angle with the inclined plane equal to the angle of friction.

$$\text{For,} \quad Q = W \frac{\sin (\alpha + \phi)}{\cos (\theta - \phi)} \quad [\text{equation (II)}].$$

Now, Q will be a *minimum* when the fraction $\frac{\sin (\alpha + \phi)}{\cos (\theta - \phi)}$ is a *minimum*. But in this fraction the numerator, $\sin (\alpha + \phi)$, is a constant quantity, since α and ϕ are supposed to be given.

$$\therefore \quad Q \text{ will be a } \textit{minimum} \text{ when } \frac{1}{\cos (\theta - \phi)} \text{ is a } \textit{minimum}.$$

$$\text{i.e., } Q \quad \quad \quad \quad \quad \quad \quad \cos (\theta - \phi) \quad \quad \quad \textit{maximum}.$$

* The student should be able to prove the results given in the above three cases independently of the general case considered in the text. (See the author's *Elementary Manual of Applied Mechanics*.)

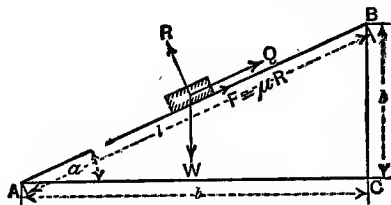
Now, the *maximum* value of a cosine is unity, and this only occurs when the angle is zero.

Hence, Q will be a *minimum* when $(\theta - \phi) = 0$.

i.e., Q " " " $\theta = \phi$.

This proves the proposition.

In what has preceded we have supposed the effort Q just able to move the body up the plane. We shall now consider the



BODY JUST SLIDING DOWN THE INCLINED PLANE.

cases where Q just prevents the body from sliding down the plane; or, when Q is employed to draw the body downwards.

Case IV.—When Q is parallel to AB and prevents the body from sliding down the plane.

In this case, by resolving along the plane, we get:—

$$Q + F = W \sin \alpha.$$

$$\therefore Q = W \sin \alpha - \mu R.$$

Resolving the forces at right angles to the plane, we get:—

$$R = W \cos \alpha.$$

$$\therefore Q = W \sin \alpha - \mu W \cos \alpha.$$

$$\text{Or, } Q = W (\sin \alpha - \mu \cos \alpha) \quad \dots \dots \dots \text{(VII)}$$

$$\text{i.e., } Q = W \frac{h - \mu b}{l} \quad \dots \dots \dots \text{(VIII)}$$

From equation (VII) it will be evident that the body will have no tendency to slide down of its own accord if:—

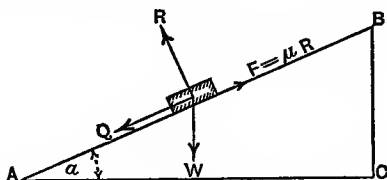
$$\sin \alpha \leq \mu \cos \alpha.$$

$$\text{i.e.—if, } \tan \alpha \leq \mu.$$

$$\text{i.e.—if, } \alpha \leq \phi \text{ (the angle of friction).}$$

When α is just slightly greater than ϕ the body would begin to slide down of its own accord, if not prevented by the force, Q . This affords a means of determining the coefficient of friction between two bodies, as explained in Lecture V.

If α is less than ϕ , then an effort, Q , must be applied to the body to drag it down the plane. Then, we get:—



FORCE REQUIRED TO PULL THE BODY DOWN THE PLANE.

$$Q + W \sin \alpha = F = \mu R = \mu W \cos \alpha,$$

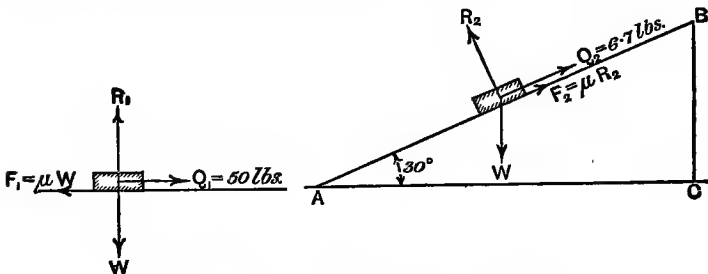
\therefore

$$Q = W (\mu \cos \alpha - \sin \alpha)$$

Or,

$$Q = W \frac{\mu b - h}{l} \quad \dots \dots (IX)$$

EXAMPLE I.—A horizontal force of 50 lbs. is just required to move a weight of W lbs. on a rough horizontal plane. If the plane be now inclined at an angle of 30° , a force of 6.7 lbs. acting parallel to the plane is required to keep the weight from sliding down. Determine the weight, W , and the coefficient of friction between the weight and the plane.



TO DETERMINE THE COEFFICIENT OF FRICTION.

ANSWER.—On the horizontal plane, we have:—

$$F_1 = \mu W = Q_1,$$

\therefore

$$\mu = \frac{50}{W} \quad \dots \dots (1)$$

Since the body is just kept from sliding down the inclined plane, both Q_2 and F_2 will act up the plane. Then:—

$$Q_2 + \mu R_2 = W \sin 30^\circ,$$

$$\therefore Q_2 = W (\sin 30^\circ - \mu \cos 30^\circ),$$

$$\therefore 6.7 = W \left(\frac{1}{2} - \mu \times \frac{\sqrt{3}}{2} \right).$$

Substituting the value of μ given in equation (1), we get:—

$$6.7 = \frac{W}{2} \left(1 - \frac{50 \times \sqrt{3}}{W} \right).$$

$$\therefore 13.4 = W - 50\sqrt{3}.$$

$$\therefore W = 13.4 + 50 \times 1.732 = 100 \text{ lbs.}$$

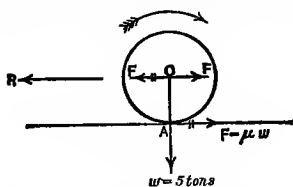
From equation (1), we get:—

$$\mu = \frac{50}{100} = .5.$$

EXAMPLE II.—Suppose a locomotive weighs 30 tons, and that the share of this weight borne by the driving wheels is 10 tons. Then, if the coefficient of friction between the wheels and the rails be .2, what load will the engine draw on the level if the required coefficient of traction be 10 lbs. per ton of train load? What load will this engine draw at the same rate up an incline of 1 in 20?

ANSWER.—(1) *On the level line.*

Let the circle represent one of the driving wheels of the locomotive, and let the wheel turn in the direction shown by



QUESTION ON TRACTION AND FRICTION.

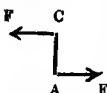
the arrow. Since there are two driving wheels the weight, w , on each will be 5 tons.

Let F = Friction between each wheel and its rail.

„ μ = Coefficient of friction between wheel and rail = .2.

Then, $F = \mu w = .2 \times (5 \times 2,240) = 2,240 \text{ lbs.}$,
and acts in the direction A F.

At O, the centre of the wheel, introduce two opposite forces, F , F , each equal and parallel to the force F at A. This will not affect the equilibrium of the system. After this has been done, it will be evident that the forces standing thus:—



form a *couple*, the moment of which is $F \times A C$. This is the couple resisting the rotation of the wheel about its centre, C, and, therefore, must be equal in moment but opposite in sign, to the couple due to the force on the crank-pin as caused by the steam pressure on the piston. In the meantime, we are concerned only with the remaining force, F , acting to the right at C. This force tends to pull the centre O, and, therefore, the whole train to the right. If R be the resistance offered by the train, then, since there are two driving wheels:—

$$R = 2 F = 2 \times 2,240 = 4,480 \text{ lbs.}$$

Let W_1 = Total weight of engine and train in tons.

Then, since the traction is 10 lbs. per ton, we get:—

$$R = 10 W_1,$$

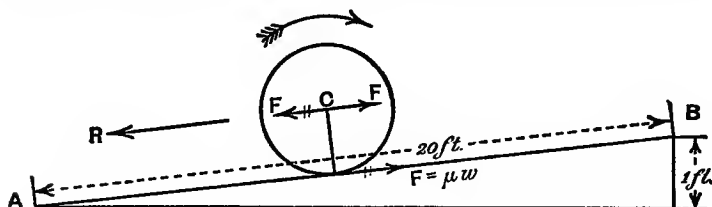
$$\therefore 10 W_1 = 4,480,$$

$$\text{i.e., } W_1 = 448 \text{ tons.}$$

The engine will, therefore, be able to draw a load of $(448 - 30)$, or 418 tons without fear of the driving wheels slipping on the rails.

(2) *On the gradient.*

Since the inclination of the rails is small (1 in 20), we may assume the pressure or reaction between the wheel and rail to be still = w tons.



QUESTION ON TRACTION AND FRICTION.

Hence, F will be the same as before, viz., 2,240 lbs.

By reasoning as in the previous case, we get:—

$$R = 2 F = 4,480 \text{ lbs.}$$

Let, W_2 = Total weight in tons of engine and train on incline.

Now, suppose the train to move from A to B, a distance of 20 feet. Then, by the *Principle of Work*, we get:—

$$\left. \begin{array}{l} \text{Total work done on in-} \\ \text{cline from A to B} \end{array} \right\} = \left\{ \begin{array}{l} \text{Work done against traction from} \\ \text{A to B.} \\ + \text{Work done against gravity from} \\ \text{A to B.} \end{array} \right.$$

$$\text{But, Total work done} = R \times 20 = 4,480 \times 20 \text{ (ft.-lbs.)}$$

$$\text{Work done against traction} = (10 \times W_2) \times 20 = 200 W_2 \text{ ft.-lbs.}$$

$$\text{" " gravity} = (W_2 \times 2,240) \times 1 = 2,240 W_2 \text{ ,,}$$

$$\therefore 4,480 \times 20 = 200 W_2 + 2,240 W_2,$$

$$\text{i.e., } W_2 = \frac{4,480 \times 20}{2,440} = 36.72 \text{ tons.}$$

Thus, the engine will only be able to draw a load of (36.72—30), or 6.72 tons up an incline of 1 in 20. Any load beyond this would cause a greater resistance than is provided for by the friction between the driving wheels and the rails.

EXAMPLE III.—What must be the effective horse-power of a locomotive engine which moves at a steady speed of 40 miles an hour on a level line, the resistance being estimated at 20 lbs. per ton, and the weight of the engine and train being 200 tons? If the engine continue to exert the same power when ascending a gradient of 1 in 100, what would be the speed?

ANSWER.—(1) *On the level line.*

$$\text{Total resistance overcome} = 200 \times 20 = 4,000 \text{ lbs.}$$

$$\text{Speed of train} = 40 \text{ miles per hour.}$$

$$\text{" " } = \frac{40 \times 5,280}{60} = 3,520 \text{ ft. per minute.}$$

$$\therefore \text{Work done per minute} = 4,000 \times 3,520 \text{ (ft.-lbs.)}$$

$$\therefore \text{H.P.} = \frac{4,000 \times 3,520}{33,000} = 426.6.$$

(2) *On the gradient.*

$$\left. \begin{array}{l} \text{Here, Total work done} \\ \text{per minute} \end{array} \right\} = \left\{ \begin{array}{l} \text{Work done against friction per} \\ \text{minute.} \\ + \text{Work done against gravity per} \\ \text{minute.} \end{array} \right.$$

Let v = Speed of train on gradient in feet per minute.

As before, Total work done = 4,000 × 3,520 (ft.-lbs. per minute).

$$\text{Work done against friction} = (200 \times 20) v \quad \text{"} \quad \text{"}$$

$$\text{Work done against gravity} = (200 \times 2,240) \times \frac{v}{100} \quad \text{"} \quad \text{"}$$

In practice, we seldom find two planes arranged as shown above. One plane only is used, and the trucks run on parallel lines of rails, being connected by a rope or chain which passes round a pulley or drum at the top of the plane. In this case, $\alpha_1 = \alpha_2 = \alpha$; $l_1 = l_2 = l$; $b_1 = b_2 = b$; and the above equations take the simple forms:—

$$\frac{W_1}{W_2} = \frac{\sin \alpha - \mu \cos \alpha}{\sin \alpha + \mu \cos \alpha} \quad \dots \quad (\text{XI})$$

Or, dividing by $\cos \alpha$,
$$\frac{W_1}{W_2} = \frac{\tan \alpha - \mu}{\tan \alpha + \mu} \quad \dots \quad (\text{XI}_a)$$

Or,
$$\frac{W_1}{W_2} = \frac{h - \mu b}{h + \mu b} \quad \dots \quad (\text{XI}_b)$$

This determines the relation between W_1 and W_2 , when W_2 is just able to draw up W_1 . If W_2 be greater than that obtained from equation (XI), the motion will be accelerated. To obtain a uniform motion for given loads, W_1 , W_2 , we must either adjust the inclination of the plane, or provide the drum with a friction brake when W_2 is greater than necessary, or with the assistance of an engine when W_2 is less than required.

(1) *To determine the requisite inclination of the plane when W_2 is required to draw up W_1 .*

From equation (XI_a), we get:—

$$W_1 \tan \alpha + \mu W_1 = W_2 \tan \alpha - \mu W_2,$$

$$\therefore \mu (W_2 + W_1) = (W_2 - W_1) \tan \alpha,$$

$$\therefore \tan \alpha = \mu \frac{W_2 + W_1}{W_2 - W_1}.$$

(2) *To determine the friction couple, which must be applied to the pulley or drum at the top of a double inclined plane, in order to obtain a uniform motion when W_2 is too great.*

This problem is very similar to the case of a driving belt transmitting motion in machinery. The tension in the two parts of the rope or chain is not now the same throughout its length, being greater on the side of the descending load W_2 .

Let Q_1 , Q_2 , denote the tension in the two parts of the rope or chain on the driven and driving sides respectively.

Then,
$$Q_2 > Q_1.$$

The following figure represents a side elevation and plan of the double inclined plane with its pulley and loads, W_1 , W_2 .

Let M = Friction couple required to be applied to the brake wheel, BW , by the brake handle, BH .

„ r = Radius of drums.

Taking moments about the centre of this wheel, we get:—

$$M + Q_1 r = Q_2 r,$$

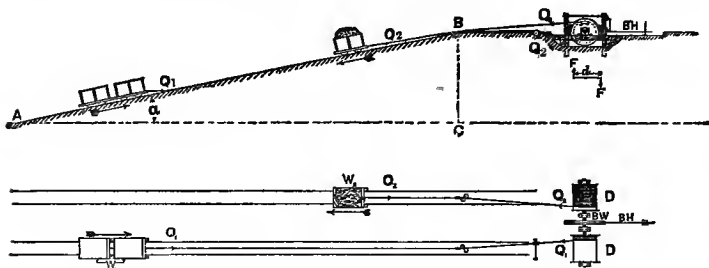
$$\therefore M = (Q_2 - Q_1) r, \quad \dots \dots \dots (1)$$

$$\text{But, } Q_2 = W_2 (\sin \alpha - \mu \cos \alpha),$$

$$\text{And, } Q_1 = W_1 (\sin \alpha + \mu \cos \alpha),$$

$$\therefore M = \left\{ (W_2 - W_1) \sin \alpha - \mu (W_2 + W_1) \cos \alpha \right\} r \quad (\text{XII})$$

$$\text{Or, } M = \frac{r}{l} \left\{ h (W_2 - W_1) - \mu b (W_2 + W_1) \right\}. \quad (\text{XII}_a)$$



PRACTICAL EXAMPLE OF THE DOUBLE INCLINED PLANE.

(3) If W_2 is not sufficient to overcome W_1 , then a moment, M , must be applied to the pulley or drum, D , to assist it.

In this case, by taking moments about the centre of the pulley as before, we get:—

$$M + Q_2 r = Q_1 r,$$

$$\therefore M = (Q_1 - Q_2) r,$$

$$\text{Hence, } M = \left\{ (W_1 - W_2) \sin \alpha + \mu (W_1 + W_2) \cos \alpha \right\} r \quad (\text{XIII})$$

$$\text{Or, } M = \frac{r}{l} \left\{ h (W_1 - W_2) + \mu b (W_1 + W_2) \right\}. \quad (\text{XIII}_a)$$

EXAMPLE IV.—In a double inclined plane having a rise of 1 in 20, the loaded and empty trucks run on parallel rails and are connected by a rope which passes over a pulley, 6 feet in diameter, at the top of the plane. Find the greatest number of empty trucks which a descending loaded one is capable of drawing up; having given weight of each empty truck, 5 cwts., weight of material in loaded truck, 20 cwts., the coefficient of traction on the level being taken at 20 lbs. per ton. Again, if 5 loaded trucks going down pull up an equal number of empty ones, what must be the mean frictional resistance on the circumference of a brake wheel, 3 feet in diameter, fitted on the pulley at the top of the incline, so that the whole may be kept moving uniformly?

ANSWER.—Let w = Weight of empty truck = 5 cwts.

„ W = Weight of material in loaded truck = 20 cwts.

Since the coefficient of traction on the level is 20 lbs. per ton,

$$\therefore \mu = \frac{20}{2,240} = \frac{1}{112}.$$

Again, since the inclination of the plane is small, we may assume:—

$$\text{That,} \quad \cos \alpha = 1 \text{ and } \sin \alpha = \frac{1}{20}.$$

Now, let there be n empty trucks drawn up by a descending loaded one.

Then, according to the previous notation:—

$$W_1 = n w = 5 n \text{ cwts.}$$

$$W_2 = W + w = 25 \text{ cwts.}$$

$$\frac{W_1}{W_2} = \frac{\sin \alpha - \mu \cos \alpha}{\sin \alpha + \mu \cos \alpha} \text{ [equation (XI)].}$$

$$\therefore \frac{5n}{25} = \frac{\frac{1}{20} - \frac{1}{112} \times 1}{\frac{1}{20} + \frac{1}{112} \times 1}.$$

$$\therefore \frac{n}{5} = \frac{23}{33}$$

$$\therefore n = \frac{115}{33} = 3 \text{ fully.}$$

Or, the greatest number of empty trucks that can be drawn up by 1 descending loaded truck is 3.

Next, with 5 loaded trucks going down and 5 empty ones coming up, we have:—

$W_1 = 5w = 25$ cwts.; $W_2 = 5(W + w) = 125$ cwts.
and $R = \text{Radius of drum} = 3$ ft.

The friction moment to be applied to the brake wheel at the top of the plane is by equation (XII):—

$$M = \left\{ (W_2 - W_1) \sin \alpha - \mu (W_2 + W_1) \cos \alpha \right\} R.$$

Substituting the above values, we get:—

$$M = \left\{ (125 - 25) \times \frac{1}{20} - \frac{1}{112} \times (125 + 25) \times 1 \right\} 3 \text{ (ft.-cwts.)}$$

$$,, = \frac{410 \times 3}{112} \text{ ft.-cwts.}$$

i.e., $M = 1,230$ ft.-lbs.

Let $F = \text{Mean frictional resistance applied at circumference of brake wheel.}$

„ $r_b = \text{Radius of brake wheel} = 1\frac{1}{2}$ feet.

Then, $\text{Friction couple} = F \times r_b = M.$

$$F \times 1\frac{1}{2} = 410 \times 3.$$

$$\therefore F = \frac{410 \times 3}{1\frac{1}{2}} = 820 \text{ lbs.}$$

EXAMPLE V.—In the latter part of Example IV., suppose the operations to be reversed, so that the five loaded trucks are to be hauled up the plane by means of an engine situated at the top of the plane, the engine being assisted by the descending five empty trucks. Find the tensions in the two parts of the hauling rope, and the H.P. of the engine; given the length of inclined plane, 1 mile and the time taken to complete the run, five minutes.

ANSWER.—From last example we get the following data:—

$W_1 = \text{Weight of five empty trucks} = 25$ cwts.,

$W_2 = \text{ „ „ loaded „} = 125$ „

$\mu = \frac{1}{112}, \sin \alpha = \frac{1}{20}, \cos \alpha = 1$, approximately.

Let $Q_1 = \text{Tension in that part of rope attached to } W_1,$
„ $Q_2 = \text{Tension „ „ „ } W_2$

Then, since W_1 is let down the plane, we get :—

$$Q_1 = W_1 (\sin \alpha - \mu \cos \alpha), \text{ [equation (VII)]}$$

$$\therefore Q_1 = 25 \left(\frac{1}{20} - \frac{1}{112} \times 1 \right) \text{ cwts.} = 115 \text{ lbs.}$$

Also, since W_2 is pulled up the plane, we get :—

$$Q_2 = W_2 (\sin \alpha + \mu \cos \alpha), \text{ [equation (II}_b\text{)]}$$

$$\therefore Q_2 = 125 \left(\frac{1}{20} + \frac{1}{112} \times 1 \right) \text{ cwts.} = 825 \text{ lbs.}$$

Let l = Length of incline = 5,280 feet.

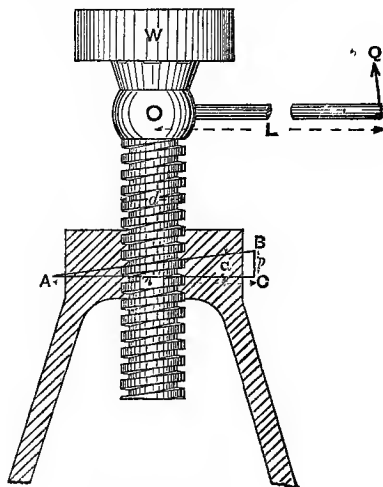
„ t = Time taken to traverse it = 5 minutes.

Then, $\left. \begin{array}{l} \text{Work done} \\ \text{by engine} \\ \text{per minute} \end{array} \right\} = \left\{ \begin{array}{l} \text{Work done per minute by rope in pulling} \\ \text{up full trucks minus work done per} \\ \text{minute on rope by descending trucks.} \end{array} \right.$

$$= \frac{l}{t} \times Q_2 - \frac{l}{t} \times Q_1 = \frac{l}{t} (Q_2 - Q_1).$$

$$\therefore \text{H.P. of engine} = \frac{\frac{l}{t} (Q_2 - Q_1)}{33,000} = \frac{5,280 \times 710}{5 \times 33,000} = 22.72.$$

Screws.—The various forms of screw threads, their development, characteristics, and manufacture, have been fully described and illustrated in our *Elementary Manual of Applied Mechanics*, and, therefore, need not be further considered here. In what follows we shall content ourselves by determining the *Advantage* and *Efficiency* of the ordinary screw arrangement. Take the case of the square threaded screw working in its nut, and suppose the pressure, due to the load, W , to be uniformly distributed along the bearing surface of the thread. Since the pitch angle, α , is everywhere the same, it will be sufficient to take a single point on the screw thread,



THE SCREW.

and consider the whole load, W , concentrated at this point. We have then the case of an inclined plane, $A B C$, as shown by the accompanying figure, which represents an ideal helix or screw line, traced on a cylinder, and a development of one complete turn of this line.

- Let p = Pitch of screw thread,
 „ d = Mean diameter of cylinder of bolt,
 „ Q = Turning effort applied to lever or spanner,
 „ L = Leverage of Q measured from the axis of the bolt,
 „ μ = Coefficient of friction between nut and screw.
 „ H = Force acting along $A C$ due to the effort, Q .

Then, by the “Principle of Moments,” we get:—

$$Q \times L = H \times \frac{d}{2}$$

Or,
$$Q = H \times \frac{d}{2L}.$$

And from equation (II.) in this Lecture:—

$$H = W \tan (\alpha + \phi).$$

$\therefore Q = W \frac{d}{2L} \tan (\alpha + \phi).$

Or,
$$\frac{Q}{W} = \frac{d}{2L} \tan (\alpha + \phi). \quad \dots \quad (XIV)$$

$\therefore \frac{Q}{W} = \frac{d}{2L} \left(\frac{\tan \alpha + \tan \phi}{1 - \tan \alpha \tan \phi} \right).$

But, from the figure,

$$\tan \alpha = \frac{B C}{A C} = \frac{p}{\pi d}, \text{ and } \tan \phi = \mu.$$

$\therefore \frac{Q}{W} = \frac{d}{2L} \left(\frac{p + \mu \pi d}{\pi d - \mu p} \right). \quad \dots \quad (XV)$

Hence,

$$\text{Actual Advantage} = \frac{W}{Q} = \frac{2L}{d} \left(\frac{\pi d - \mu p}{p + \mu \pi d} \right). \quad \dots \quad (XVI)$$

Efficiency of Screw.—Suppose the weight to be raised through a distance equal to the pitch, p . Then Q will have moved through a distance equal to $2 \pi L$. Hence:—

$$\text{Efficiency} = \frac{W \times p}{Q \times 2 \pi L}.$$

$$= \frac{2 L}{d} \left(\frac{1}{\tan (\alpha + \phi)} \times \frac{p}{2 \pi L} \right)$$

$$= \frac{1}{\tan (\alpha + \phi)} \frac{p}{\pi d},$$

$$\text{i.e., Efficiency} = \frac{\tan \alpha}{\tan (\alpha + \phi)} \quad \dots \dots \dots \text{(XVII)}$$

Maximum Efficiency of Screw.—We can now find what value of α will give the *greatest efficiency*. Clearly the efficiency will be a *maximum* when $\frac{\tan \alpha}{\tan (\alpha + \phi)}$ is a *maximum*.

$$\text{But, } \frac{\tan \alpha}{\tan (\alpha + \phi)} = \frac{\sin \alpha \cos (\alpha + \phi)}{\cos \alpha \sin (\alpha + \phi)},$$

$$= \frac{\sin (2 \alpha + \phi) - \sin \phi}{\sin (2 \alpha + \phi) + \sin \phi},$$

$$= 1 - \frac{2 \sin \phi}{\sin (2 \alpha + \phi) + \sin \phi}.$$

From this it is clear that the efficiency will be a *maximum* when $\frac{2 \sin \phi}{\sin (2 \alpha + \phi) + \sin \phi}$ is a *minimum*.

i.e., when $\frac{1}{\sin (2 \alpha + \phi) + \sin \phi}$ is a *minimum*.

i.e., when $\sin (2 \alpha + \phi)$ is a *maximum*.

But the greatest value for the sine of an angle is unity, and this occurs when the angle is 90° .

The efficiency will, therefore, be a *maximum* when :—

$$2 \alpha + \phi = 90^\circ.$$

$$\text{Or, } \alpha = 45^\circ - \frac{\phi}{2}.$$

Substituting this value for α in the expression for the efficiency, we get :—

$$\begin{aligned}\text{Maximum Efficiency} &= \frac{\tan \left(45^\circ - \frac{\phi}{2}\right)}{\tan \left(45^\circ + \frac{\phi}{2}\right)} \\&= \frac{1 - \tan \frac{\phi}{2}}{1 + \tan \frac{\phi}{2}} \div \frac{1 + \tan \frac{\phi}{2}}{1 - \tan \frac{\phi}{2}}, \\&= \left(\frac{1 - \tan \frac{1}{2} \phi}{1 + \tan \frac{1}{2} \phi}\right)^2.\end{aligned}$$

But ϕ is always a small angle, and we may, therefore, substitute $\frac{1}{2} \mu$ for $\tan \frac{1}{2} \phi$, so that:—

$$\text{Maximum Efficiency} = \left(\frac{1 - \frac{1}{2} \mu}{1 + \frac{1}{2} \mu}\right)^2 \text{ (approximately).}$$

From this we see that for the maximum efficiency in the case of a screw the best pitch angle is 45° nearly.*

Taking the coefficient of friction = .16, and pitch angle 45° , we get:—

$$\text{Maximum Efficiency} = \left(\frac{1 - \frac{1}{2} \times .16}{1 + \frac{1}{2} \times .16}\right)^2 = .72 \text{ or } 72 \text{ per cent. nearly.}^*$$

If the pitch angle be greater than 45° , it will be *always* possible to reverse the action of the screw, *quite irrespective of the coefficient of friction*; or, a downward pressure, W , will cause the screw to revolve. Instances of this may be met with in some forms of hand drills, and in certain instruments used for domestic purposes. The student will be able, from the general investigations in Lecture VII., to prove that the efficiency of a screw when working in the reversed way is given by the equation.

$$\text{Reversed Efficiency} = \frac{\tan (\alpha - \phi)}{\tan \alpha}.$$

Non-reversibility of Ordinary Screws and Nuts.—In bolts and most other applications of screws the pitch angle is very much less than 45° , consequently, the efficiency of these screws is often

* By the introduction of hardened steel friction balls between the threads of the screw and nut, an efficiency of over 90 per cent. may be obtained. In practice, it is found with ordinary screws, that the best *all round* results are obtained with a pitch angle of about 20° . The screw will reverse at a less angle than 45° , in the case of well lubricated worm gear, if the efficiency is well over 50 per cent (See Lecture X.)

lower than 20 per cent. In such cases, however, mechanical advantage and non-reversibility are the objects chiefly aimed at, and not high efficiency.

Tension in Bolts due to Screwing Up.—Consider the case of a square-threaded screw.

- Let W = Tension in bolt due to screwing up,
 „ Q = Force applied at end of spanner,
 „ L = Length of spanner,
 „ d = Mean diameter of bolt thread,
 „ p = Pitch of thread,
 „ μ = Coefficient of friction between screw and its nut,
 „ μ_1 = „ „ „ nut and its washer.

Then, *Friction between nut and washer* = $\mu_1 W$.

Suppose this friction to act at the circumference of a circle of diameter D ; in other words, let D be the diameter of the friction circle between nut and washer.

Then, *Friction moment between nut and washer* = $\mu_1 W \times \frac{D}{2}$.

Hence, by taking moments about the axis of the bolt, we get:—

$$Q \times L = H \times \frac{d}{2} + \mu_1 W \times \frac{D}{2}$$

$$” = \frac{W}{2} \left\{ \frac{p + \mu \pi d}{\pi d - \mu p} \times d + \mu_1 D \right\}$$

[From previous formula for H and equation XV.]

$$\therefore W = \frac{2 Q L}{\frac{p + \mu \pi d}{\pi d - \mu p} d + \mu_1 D}.$$

The average length of a spanner is $L = 15 d$, and D may be taken at $\frac{4}{3} d$, while $\mu_1 = \mu$ very nearly.

$$\text{Hence, Tension in bolt} = W = \frac{90 (\pi d - \mu p)}{7 \mu \pi d + (3 - 4 \mu^2) p} Q.$$

For ordinary sized bolts we may take $\mu = 0.15$, and $p = 0.16 d$.

Hence, substituting these values in the last equation, we get:—

$$\text{Tension in bolt} = \frac{90 \times 3.1176}{3.76} Q = 75 Q, \text{ nearly.}$$

Now, suppose a force of 30 lbs. to be applied at the end of the spanner by a workman, then :—

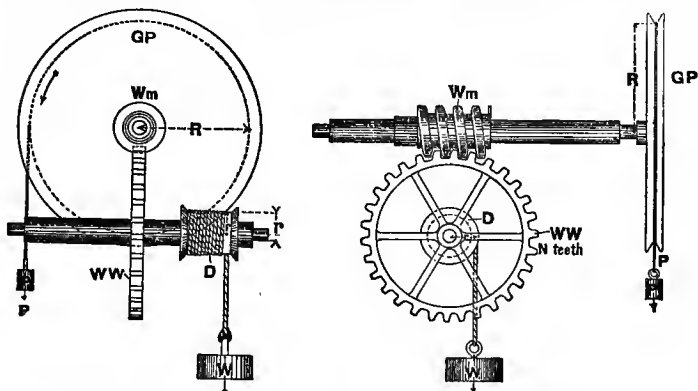
$$\text{Tension in bolt} = 75 \times 30 = 2,250 \text{ lbs.,}$$

This tension would be about sufficient to break a wrought-iron bolt $\frac{3}{8}$ inch in diameter, and would seriously injure a bolt $\frac{1}{2}$ inch in diameter.

Hence the practical rule :—*That bolts less than $\frac{3}{4}$ inch in diameter should never be employed for joints requiring to be tightly screwed up.*

In estimating the friction of such machines as screw jacks, where the end of the screw terminates in a loose cap supporting the load, the friction between the cap and the part of the screw supporting it must not be neglected. In many cases this friction is about as great as that between the thread of the screw and its nut.

EXAMPLE VI.—Apply the *Principle of Work* to calculate the relation of the effort, P, to the resistance, W, in the following



END VIEW.

SIDE VIEW.

PULLEY, WORM, WORM-WHEEL, AND WINCH DRUM.

INDEX TO PARTS.

GP represents Grooved Pulley.
Wm ,, Worm or endless
 screw.

WW represents Worm-wheel.
D ,, Drum.

combination :—In a model to show the action of an endless screw and worm-wheel, the pulley which turns the screw is 18 inches in diameter, the screw is double threaded, and the worm-wheel

has 30 teeth. On the axis of the worm-wheel is a drum $4\frac{1}{2}$ inches in diameter round which the cord is coiled. What load, W , hanging on this cord would be supported by a weight, P , of 14 lbs. at the circumference of the pulley, friction being neglected?

ANSWER.—Two views of the essential parts of this combination are given above.

Let R = Radius of pulley = 9 inches.

„ r = „ drum = $2\frac{1}{2}$ „

„ N = Number of teeth on worm-wheel = 30.

„ n = „ threads on endless screw = 2.

If the effort, P , receive a displacement equal to the circumference of the pulley, then the worm-wheel will make $\frac{n}{N}$ of a turn, since every complete turn of the worm displaces n teeth on the worm-wheel. Hence:—

$$\text{Displacement of } W = \frac{n}{N} \times 2\pi r.$$

But, by the *Principle of Work*, we have:—

$$P \times \text{its displacement} = W \times \text{its displacement}.$$

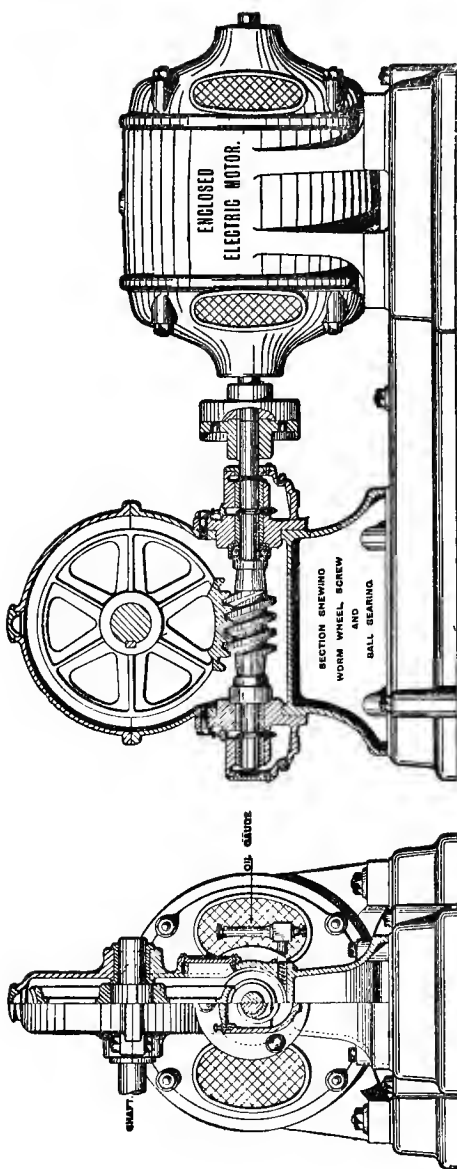
$$\therefore P \times 2\pi R = W \times \frac{n}{N} \times 2\pi r.$$

$$\therefore \frac{P}{W} = \frac{n r}{N R}.$$

Substituting the values of P , n , N , r and R , we get:—

$$\frac{14}{W} = \frac{2 \times 2\frac{1}{2}}{30 \times 9}.$$

$$\therefore W = 840 \text{ lbs.}$$



ELECTROMOTOR WITH SELF-OILING, SPEED REDUCING WORM GEAR, by David Brown & Sons, Huddersfield.

Speed Reducing Worm Gear.—As a practical illustration of the previous example, the following figures show a design which is now much used, where the high speeds of Motors must be reduced for the working of elevators, pumps, cranes, and other kinds of similar machines.

The gear consists of a phosphor bronze worm wheel, driven by a forged steel single or multiple-threaded worm, completely enclosed in a cast-iron case. The worm and lower periphery of its wheel, as well as the ball bearings, run in an oil bath. In order that this bath may be duly inspected and fed without opening of the cast-iron case, it is fitted with an oil sight feed gauge. This arrangement ensures silent transmission of power at a much higher efficiency than dry worm gearing with ordinary bearings. (*See Index for special description and other applications of ball bearings.*)

LECTURE IX.—QUESTIONS.

1. State and prove the relation between the weight, W , of a body resting on a rough inclined plane, the reaction, R , from the plane, and the force, Q , necessary to just balance the weight: (1) when the force, Q , acts parallel to the plane, (2) when it acts parallel to the base, (3) when it acts at an angle, θ , to the plane.

2. The resistance of friction along an inclined plane is taken at 150 lbs. for each ton of weight moved. Find the work done in drawing 2 tons up 100 feet of an incline which rises 1 foot in height for 25 in length. *Ans.* 47,920 ft.-lbs.

3. If 150 lbs. per ton is a sufficient tractive force to draw a loaded waggon along a horizontal road, what tractive force per ton will be required to draw the load up an incline 1 in 10? *Ans.* 374 lbs. per ton.

4. What must be the effective horse-power of a locomotive engine which moves at a steady speed of 40 miles per hour on a level rail, the resistance being 15 lbs. per ton, and the weight of the engine and train being 100 tons? If the rails were laid at a gradient of 1 in 100, what additional horse-power would be required? *Ans.* 160 H.P.; 238.93 H.P.

5. A train of 200 tons ascends an incline which has a rise of .5 foot per cent. (i.e., 5 feet in 1,000), with a uniform speed of 30 miles per hour, what is the effective horse-power of the engine, the friction being 5.5 lbs. to the ton? *Ans.* 267.2 H.P.

6. A train of 330 tons ascends an incline which has a rise of .2 per cent. (i.e., 2 feet in 1,000), what is the maximum speed in miles per hour with an engine of 120 horse-power, the friction being 8 lbs. to the ton? *Ans.* 10.92 miles per hour.

7. Prove the formula for the relation between the weights, W_1 and W_2 , in the double inclined plane, taking friction into account. *Ex.* A double inclined plane is formed by two inclined planes placed back to back so that they have a common summit. Their inclinations to the horizon are 30° and 40° respectively. The weight of the body on the latter plane is 500 lbs.; find the greatest weight which it is capable of drawing up on the other plane, coefficient of friction in both cases being 0.2. *Ans.* 363.9 lbs.

8. A stationary engine at the top of an inclined plane is employed to draw loaded waggons up the plane, and is assisted by an equal number of empty waggons which descend by a parallel line of rails. The total weight of loaded waggons is 35 tons, the weight of the descending empty ones being 12 tons. Find the maximum H.P. developed by the engine, the maximum speed of the waggons being 6 miles per hour; inclination of plane 1 in 15; coefficient of traction on the level being taken at 10 lbs. per ton. Find also the tensions in the two ropes. *Ans.* 62.48 H.P.; 5,577 lbs.; 1,672 lbs.

9. Define the pitch of a screw. In the Whitworth angular screw-thread, what is the angle made by opposite sides of the thread? To what extent is the thread rounded off at the top and bottom? Distinguish between a *single* and a *double-threaded* screw; in what cases would the latter be used? Why are holding down bolts made with angular threads?

10. What is meant by *backlash*? How may backlash be prevented in a screw?

11. Sketch and describe some form of screw-jack, and estimate the relation between the force applied and the resistance overcome, friction being neglected.

12. Find the relation between Q and W in an ordinary screw-jack fitted with a square-threaded screw, taking friction into account. In an ordinary screw-jack the mean diameter of the screw-thread is 4 inches, pitch of screw 1 inch, length of lever measured from axis of screw 4 feet; find the weight raised by an effort of 60 lbs., applied at the end of the lever, the coefficient of friction being taken at 0.1. *Ans.* 3.55 tons.

13. Find an expression for the efficiency of the screw in last question, and state its numerical value for example given. What are the conditions for maximum efficiency? Prove your answer. *Ans.* 44 per cent.

14. In question 12 find the weight raised and the efficiency of the apparatus when the friction between the cylinder of the screw and the loose cap fitted thereon is taken into account. You may take diameter of friction circle for loose cap equal to mean diameter of screw thread, and coefficient of friction same as before. *Ans.* 1.57 tons; 19.4 per cent.

15. State the principle of work, and apply it to calculate the relation of the force, P , to the resistance, W , in the following combination:—*Ex.* A worm-wheel having 16 teeth forms the nut of a screw of $\frac{1}{2}$ -inch pitch; an endless screw, actuated by a lever handle of 14 inches in length, works in the worm-wheel. Find the pressure exerted by the screw when a force of 20 lbs. is applied to the end of the lever handle. *Ans.* 25 tons.

16. Explain the mechanical advantage resulting from the employment of an endless screw and worm-wheel. The lever handle which turns an endless screw is 14 inches long, the worm-wheel has 32 teeth, and a weight, W , hangs by a cord from a drum of 6 inches in diameter, whose axis coincides with that of the worm-wheel. If a pressure, P , be applied to the lever handle, find the ratio of P to W for equilibrium. *Ans.* 3 : 448.

17. Sketch and describe the screw lifting jack as fitted with screw and worm-wheel gear. The handle being 15 inches long, the pitch of the screw $1\frac{1}{4}$ inches, and the worm-wheel having 15 teeth, find the force on the handle for raising 5 tons (friction is neglected). *Ans.* 9.9 lbs.

18. A common screw-jack, with a lever 16 inches in length, has a worm-wheel of 20 teeth, and a screw of $1\frac{1}{2}$ inches pitch. Sketch the arrangement and calculate the weight lifted by the application of a constant pressure of 30 lbs. at the end of the handle, friction being neglected. *Ans.* 48,255 lbs.

19. Describe, with sketches, a common screw-jack. If the radius at which the hand acts is 15 inches, and the pitch of the screw is 0.3 inch, what is the velocity ratio? How would you proceed to find its mechanical advantage experimentally under various loads? What sort of results would you expect to find?

20. A screw-jack has a screw with a pitch of $\frac{3}{4}$ inch, and the turning force is applied at the end of two bars, the effective length of each measured from the axis of the screw being 33 inches. Assuming that the loss in friction of the screw amounts to 62 per cent. of the whole work they do, what load could be lifted by this jack by two men, each applying a force of 45 lbs. at the end of the bar he works? *Ans.* 4.2 tons. (C. & G., 1901, O., Sec. A.)

B.—See Appendices B and C for other questions and answers.

LECTURE IX.—A.M. INST. C.E. EXAM. QUESTIONS.

1. A force acting at an angle of 30° with an inclined plane pulls a weight of 1,000 pounds for a distance of 100 feet up the plane, which makes an angle of 30° with the horizontal. If the coefficient of friction is 0.2, find the work done by the force. (I.C.E., Oct., 1899.)

2. A wedge-shaped body rests with its hypotenuse on a rough inclined plane, its upper surface is horizontal, and its third face, which is vertical, rests against a rough vertical plane. If the wedge is on the point of motion, find the pressure, in amount and direction, that it will exert on the vertical plane. (I.C.E., Feb., 1900.)

3. Calculate what incline a vehicle of 3 tons, propelled by an engine exerting 10 H.P., will climb, taking the frictional resistance to be 60 lbs. per ton. (I.C.E., Feb., 1900.)

4. A square-threaded screw is $2\frac{1}{8}$ inches mean diameter and 1 inch in pitch. Taking the coefficient of friction as 0.1, show that the efficiency is about 0.6. (I.C.E., Feb., 1901.)

5. A locomotive weighs 35 tons, the coefficient of friction between wheels and rail is 0.18. Find the greatest pull which the engine can exert in driving itself and a train. What is the total weight of itself and train, which it can draw up an incline of 1 in 100, if the resistance to motion is 20 lbs. per ton on the level? (I.C.E., Oct., 1901.)

6. In a screw-jack, the screw makes 20 turns in raising the load 12 inches, and the length of the arm at the end of which the force is applied is 2 feet. Find the force required to lift a load of 5 cwts., the mechanical efficiency being 40 per cent. What is the "velocity ratio" and the "mechanical advantage" of the machine? (I.C.E., Oct., 1901.)

7. Find the mechanical advantage of the screw, friction being neglected. Find the tension set up in a stay by a force of 49 lbs. applied at the end of a bar 2 feet long acting on a double screw consisting of a right-handed screw of 5 threads to the inch and a left-handed screw of 6 threads to the inch. (I.C.E., Oct., 1902.)

8. A nut is moved forward by the revolution of a screw. If α be the pitch angle of the screw and β the angle of friction, show that the efficiency is $\frac{\tan \alpha}{\tan (\alpha + \beta)}$. If the nut, by pressing against the screw in the line of its axis, cause it to revolve, what is then its efficiency? Hence, find the greatest pitch a 2-inch screw may have in order that it may not be reversible. (I.C.E., Oct., 1902.)

9. If it take 600 useful H.P. to draw a train of 335 tons up a gradient of 1 in 264 at a uniform speed of 40 miles an hour, estimate the resistance per ton other than that due to ascending against gravity, and deduce the uniform speed on the level when developing the above power. (I.C.E., Feb., 1904.)

10. Find the efficiency of a screw-jack of $\frac{3}{8}$ -inch pitch and mean diameter of thread 2 inches, the coefficient of friction of the screw thread being $\frac{1}{4}$. (I.C.E., Feb., 1904.)

11. An iron wedge used for splitting a tree is struck so as to be subject to a vertical force of 4 tons. The taper of the wedge is 1 inch per foot, and the coefficient of friction against the tree is 0.1. Find the horizontal splitting force, and find the efficiency. (I.C.E., Oct., 1904.)

N.B.—See Appendices B and C for other questions and answers.

LECTURE X.

CONTENTS. — Definitions — Motion — Velocity — Acceleration — Graphical Methods — Velocity Diagrams — Falling Bodies — General Formulæ — Rotation — Angular Velocity — Circular Measure — Angular Acceleration — Composition and Resolution of Velocities — Parallelogram of Velocities — Triangle of Velocities — Polygon of Velocities — Rectangular Resolution — Composition and Resolution of Accelerations — The Hodograph — Hodograph for Motion in a Circle — Examples I., II., III., and IV. — Instantaneous Centre — Varying Velocities — Questions.

DEFINITION.—A body is said to be in Motion when it is continually changing its position in space, and to be at Rest when it retains a fixed position in space.

These are the definitions of *absolute* motion and *absolute* rest. We can never know the absolute motion of any body because we know no fixed bodies to which we may refer its positions at different times. We, therefore, can only deal with the *relative motion* of a body.

DEFINITION.—A body is said to have Relative Motion with respect to another body when it is continually changing its position relatively to that body.

Thus, take the case of a train moving on a railway. We always consider its motion relatively to some part of the earth's surface. But the train is carried round the earth's axis and also round the sun by the rotation of the earth itself. And this is not all, for we have reason to believe that the sun itself is not fixed in space but is in motion. A passenger in the train might be at rest relative to the train but he would be in motion relatively to the houses, trees, &c., which the train passed on its way.

Motions of Translation and Rotation.—The motion of a body may be either *Translatory* or *Rotary*.

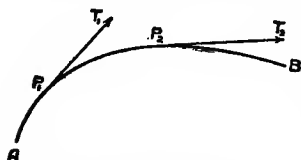
A body is said to have a motion of *simple translation* when all points in the body move with the same velocity and in the same direction at the same instant, so that no line in the body changes its direction. Hence, the motion of the whole body is known when that of any point in it is known.

A body is said to have a motion of *simple rotation* when the various points in the body describe circles about some fixed axis either within or without the body. Hence, the motion of the whole is known when that of any *line* in the body (other than the axis about which the motion takes place) is known.

The motion of a body may be *complex*; being composed or compounded of motions of translation and rotation. Thus, the connecting-rod of an engine has a complex motion. It has a motion of translation in a vertical plane containing the centre line of the engine, and a motion of rotation in the same plane about the crosshead pin.

DEFINITION.—The Path of a moving point is the line, straight or curved, which passes through all the successive positions of the point.

Direction of Motion.—The direction of motion of a body is, at any particular instant, the tangent to the path of the body at that instant, or the path itself if the motion is rectilinear.



ILLUSTRATING DIRECTION OF MOTION.

Thus, let AB be the path of a moving body. When the body occupies the position, P_1 , its direction of motion is along P_1 , T_1 , the tangent to the path at that point. Similarly, when the body occupies the position P_2 , its direction of motion is along the tangent P_2 , T_2 .

Hence, when a body moves in a circular path its direction of

motion at any instant will be perpendicular to the radius drawn to its position on the circle at that instant.

DEFINITION.—The Velocity of a body is the rate at which it changes its position.

A *velocity* is completely specified when we know (1) its *direction*, and (2) its *magnitude*.

Hence, a velocity can be completely represented by a straight line of finite length with a suitably-directed arrow head.

DEFINITION.—A body is said to be moving with Uniform Velocity when it is moving in a constant direction and passes over equal distances in equal intervals of time, however small these may be.

The last clause in the above definition is necessary, because a body *might* describe equal distances in equal times, and yet its motion might not be uniform. Thus, a train may describe 20 miles in each of two consecutive hours, and yet its motion may have varied continuously during that time; sometimes its velocity may be 60 miles an hour, and at other times it may be nil.

Uniform Velocity, how Measured.—When uniform, the velocity of a body is measured by its displacement in unit time. Thus:—

$$\text{Velocity} = \frac{\text{Displacement}}{\text{Time}}.$$

DEFINITION.—A body is said to have Unit Velocity when it describes unit distance in unit time.

The unit of distance in this country is the *foot*, and the unit of time is usually the *second*, although engineers often take the *minute*, or even the *hour*, as the unit of time. For example, the speed of a railway train is always spoken of as so many *miles per hour*, and that of the piston of an engine as so many *feet per minute*.

Whatever units may be used, we get:—

$$v = \frac{s}{t} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \dots \dots \dots (I)$$

Or,

$$s = vt$$

Where, s = Displacement, or distance described, in time, t .

And, v = Velocity, supposed to be uniform.

* [From the above definition and equation it is evident that v must be the same however small t may be. Thus, let the displacement be very small, say Δs , then the time taken to describe it will be correspondingly small, say Δt , and we get:—

$$v = \frac{\Delta s}{\Delta t}.$$

This being true for the smallest fraction of time, it must also be true in the limit.

$$\therefore \quad v = \frac{ds}{dt} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \dots \dots \dots (II)]$$

Or,

$$ds = v dt$$

DEFINITION.—A body is said to be moving with Variable Velocity when it is either changing its direction of motion or passing over unequal distances in equal intervals of time.

* Students who have no knowledge of the notation of the *Calculus*, and those merely reading for examination in the Advanced Stage of this subject, may omit for the present the text within the brackets, thus [].

From this definition it appears that a body has a *variable velocity* when the direction or magnitude of its velocity is variable. Thus, a point on the rim of the flywheel of an engine has a variable velocity whether the rotary motion of the wheel be uniform or not.

This follows at once from the fact that a velocity is only completely specified when we know its *direction* and *magnitude*, and a change in either the direction or in the magnitude causes a change in the velocity. It is usual, however, in most problems, to speak of the velocity as being uniform or variable, according as the *magnitude* of the velocity is uniform or variable.

Variable Velocity, how Measured.—When variable, the velocity of a body is measured at any particular instant by the displacement which the body would have received if it moved for a unit of time with the same velocity which it had at the instant under consideration.

Thus, we see a train approaching a station and say that its velocity is 10 miles an hour, although we at the same time observe that its velocity is diminishing rapidly, and will soon be zero. By the expression "10 miles an hour" we, therefore, do not mean that it will run 10 miles during the next hour, but simply that *if* the train continued to run for one hour with the same speed that it had at the instant the remark was made, it would travel a distance of 10 miles.

Average Velocity.—When the velocity of a body is variable, and we know its magnitudes for several positions of the body, then its *average* velocity can be found in the same way as we find the average of a series of numbers.

Thus, let $v_1, v_2, v_3, \dots, v_n$ denote the velocities at n different points in its path; then:—

$$\text{Average velocity} = \bar{v} = \frac{v_1 + v_2 + v_3 + \dots + v_n}{n}.$$

Or it may be defined as follows:—

DEFINITION.—When a body moves through a certain distance with a variable velocity, its average velocity is that uniform velocity which it would require to have in order to traverse the same distance in the same time.

$$\left. \begin{array}{l} \text{Therefore,} \\ \text{Or,} \end{array} \right\} \begin{array}{l} \bar{v} = \frac{s}{t} \\ s = \bar{v} t \end{array} \quad \dots \dots \dots (I_a)$$

If the velocity increase or decrease uniformly, then the mean or average velocity is half the sum of the initial and final velocities.

$$\text{Or,} \quad \bar{v} = \frac{v_1 + v_2}{2} \quad \dots \dots \dots (III)$$

$$\therefore s = \frac{v_1 + v_2}{2} \times t.$$

$$\text{But, } v_2 = v_1 + a t,$$

$$\therefore s = \frac{v_1 + (v_1 + a t)}{2} \times t,$$

$$\text{Or, } s = v_1 t + \frac{1}{2} a t^2. \quad \dots \quad (V)$$

In many problems the time, t , is not given, and we require to find one of the four quantities, s , a , v_1 , v_2 , having given the other three. From equations (IV) and (V) the following relation between these quantities can easily be deduced by eliminating t . Thus:—

$$\text{From equation (IV), } a = \frac{v_2 - v_1}{t},$$

$$\text{From equation (V), } s = \frac{v_1 + v_2}{2} \times t.$$

Multiplying together the corresponding sides of these equations and equating the products, we get:—

$$a s = \frac{v_2^2 - v_1^2}{2},$$

$$\therefore \left. \begin{aligned} v_2^2 - v_1^2 &= 2 a s \\ \text{Or, } v_2^2 &= v_1^2 + 2 a s \end{aligned} \right\} \dots \dots \dots (VI)$$

The above formulæ are true for all cases of *uniformly increasing* or *uniformly decreasing* velocity; but in the latter case, the acceleration will be negative, and a must be preceded by the *minus* sign.*

If the body start from rest, that is, if the time, t , be reckoned from the commencement of the motion, then, the initial velocity, $v_1 = 0$, and we get, from the above equations:—

$$v = a t \quad \dots \dots \dots (IV_a)$$

$$s = \frac{1}{2} a t^2 \quad \dots \dots \dots (V_a)$$

$$v^2 = 2 a s \quad \dots \dots \dots (VI_a)$$

Where v = velocity at end of time, t .†

* There is no need for deducing, or even stating, the corresponding formulæ when the acceleration is negative. The fewer formulæ to be committed to memory the better, and the student should learn to distinguish between positive and negative (increasing or decreasing) acceleration as indicated by difference in sign, and to supply the proper sign where necessary.

† The general formulæ (IV), (V), and (VI) should be used in all cases. When the body starts from rest, substitute $v_1 = 0$.

Graphical Methods.—Equations (III) and (IV) can be very easily represented by means of a diagram. We may here remark that diagrams of velocities, accelerations, &c., are very useful in assisting the student to answer many problems on the motion of a body, and in what follows we shall have several instances of their use when dealing with the moving parts of engines. Before explaining the following diagrams, it is necessary to remind the student that a velocity, or an acceleration, can be completely represented by a straight line. We have already seen that a *velocity* may be represented by a finite straight line. But an *acceleration* is a change of velocity per unit time. Hence, an acceleration may also be represented by a finite straight line. In the meantime, we are not concerned with the *direction* of the velocity or acceleration, so that the lines representing these may be drawn in any convenient direction.

Velocity and acceleration diagrams are constructed in a way similar to those representing work, viz., by drawing two axes at right angles, along which the velocities or accelerations and intervals of time may be plotted.

Diagram for Uniform Velocity.—Let v = velocity, supposed to be uniform, and t = time. Draw the line AB , along which intervals of time have to be plotted. Thus, let AB represent t . From A , set up AC at right angles to AB , and let AC represent the velocity, v . Complete the rectangle $ABDC$. Then, clearly, the area of $ABDC$ represents the displacement during the time, t . Thus:—

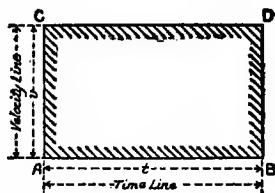


DIAGRAM FOR UNIFORM VELOCITY.

The area, $ABDC$, represents the displacement in time, t .

$$\therefore s = vt.$$

$$\begin{aligned} \text{Displacement} &= s = vt \\ &= \text{area } ABDC. \end{aligned}$$

Diagram for Uniformly Increasing Velocity.—Let a = the acceleration, and v = velocity at the end of time, t ; the initial velocity being zero. As before, let AB represent the interval of time, t . At B , the end of interval t , draw BC to represent v , and join AC . Then, as before, the area of triangle ABC represents the displacement during time, t ; since,

$$\begin{aligned} \text{Displacement} &= s = \text{mean velocity} \times \text{time} = \frac{1}{2} v \times t \\ &= \frac{1}{2} BC \times AB = \text{area } ABC. \end{aligned}$$

The velocity at any other time can be found by drawing the ordinate from the point on AB representing the given instant.

Thus, suppose AB represents 4 seconds. Then, the velocity at the end of 3 seconds from the beginning of the motion is represented by the ordinate 3 F . Similarly, at the end of the first second, the velocity is represented by the ordinate 1 D . But in this case, the velocity at the end of the first second is a measure of the acceleration; therefore, 1 D represents the acceleration.

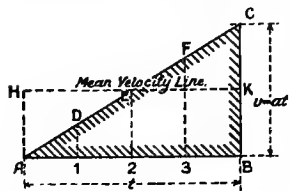


DIAGRAM FOR VELOCITY INCREASING UNIFORMLY FROM 0 TO v .

The area, ABC , represents the displacement in time, t .

$$\therefore s = \frac{1}{2} v t.$$

Join AD and produce it.

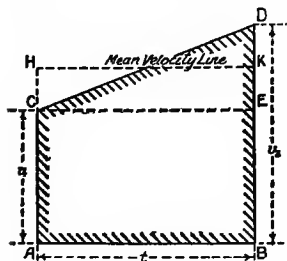


DIAGRAM FOR VELOCITY INCREASING UNIFORMLY FROM v_1 TO v_2 .

The area, $ABDC$, represents the displacement in time, t .

$$\therefore s = v_1 t + \frac{1}{2} a t^2.$$

$$\text{Here, Displacement} = s = \frac{1}{2} (v_1 + v_2) \times t = \frac{1}{2} (AC + BD) \times AB \\ = \text{area } ABDC.$$

$$\text{Also, } ED = \text{Change of velocity in time, } t = a t.$$

$$\text{And, } BD = BE + ED = v_1 + a t.$$

$$\therefore s = \frac{1}{2} (v_1 + v_1 + a t) \times t = v_1 t + \frac{1}{2} a t^2.$$

We have not drawn the corresponding diagrams for the case when the acceleration is *negative*, but the student should have

If the acceleration be given instead of the final velocity, v , then the diagram can be set out in the following manner:—

Let Al represent a unit of time. Draw lD at right angles to AB to represent the acceleration, a . Then AC is the velocity line. From this it will be seen that $v = BC = a t$.

$$\therefore s = \frac{1}{2} a t^2$$

$$,, = \frac{1}{2} a t \times t$$

$$,, = \frac{1}{2} BC \times AB$$

$$,, = \text{area } ABC.$$

We can now give similar expressions for the acceleration when this varies according to any law whatever. Thus, at any instant of time let the velocity of the body be v , and at the end of an interval of time, Δt , let it be $v + \Delta v$, then :—

$$\begin{aligned} \text{Acceleration} &= \frac{\text{Change of velocity}}{\text{Time required}}, \\ \text{,,} &= \frac{(v + \Delta v) - v}{\Delta t} = \frac{\Delta v}{\Delta t}. \end{aligned}$$

This being true, however small Δt may be, it is, therefore, true in the limit, hence :—

$$a = \frac{dv}{dt}.$$

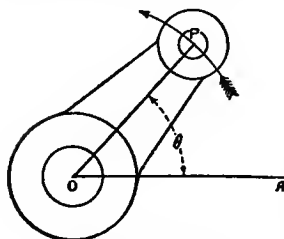
But,

$$v = \frac{ds}{dt},$$

\therefore

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}. \quad \dots \dots \dots \text{(VII)]}$$

Body Rotating about an Axis.—Angular Velocity.—We have already said that the motion of a body rotating about an axis is



TO ILLUSTRATE ANGULAR VELOCITY.

completely known when that of any line in the body, other than the axis of rotation, is known. It is most convenient to take this line passing through the axis, and perpendicular to it. Thus, let O be the intersection of the axis with the plane of the paper, OP a line in the body perpendicular to the axis through O . Then the motion of the body is known when that of the line OP is known. The motion of the line OP is measured by the angle which

it describes round the point, O , in unit time. This angle is then spoken of as the **angular velocity** of the body. Hence the following :—

DEFINITION.—The angular velocity of a body about an axis is the rate of the angular displacement of any line in the body perpendicular to that axis.

Angular velocity, like linear velocity, may be either *uniform* or *variable*, according as equal or unequal angles are described in equal intervals of time.

Uniform Angular Velocity.—Let the centre line, OP , of the crank in the above figure sweep out the angle, $\angle AOP = \theta$, in the interval of time, t ; then the angular velocity of the body (usually denoted by the Greek letter ω) is :—

$$\omega = \frac{\theta}{t} \quad \dots \dots \dots \text{(VIII)}$$

The angle, θ , is measured in *circular units*, and not in degrees. The unit angle in circular measure is called the *radian*, and may be defined as *the angle subtended at the centre of a circle by an arc of its circumference, equal in length to the radius of the circle*. Hence, if t is in seconds, the *unit of angular velocity* will be the *radian per second*.

Since the length of the arc subtending a right angle is $\frac{\pi}{2} \times r$, and, therefore, the circular measure of a right angle equal to $\frac{\pi}{2}$ radians, we may easily determine the number of *degrees* in a radian. Thus :—

$$\text{Degrees in 1 radian : Degrees in 1 right angle} = 1 : \frac{\pi}{2}.$$

$$\therefore \quad \text{Degrees in 1 radian} = \frac{90^\circ}{\frac{\pi}{2}} = \frac{180^\circ}{3.1416} = 57.29.$$

In general, if ρ be radians or the circular measure of an angle of θ° , then :—

$$\text{Since,} \quad 90^\circ : \theta^\circ :: \frac{\pi}{2} : \rho,$$

$$\text{We get,} \quad \rho = \frac{\theta^\circ \pi}{180} \text{ radians.}$$

Hence, if the angle described in time, t , by OP , be θ° , we get :—

$$\omega = \frac{\theta^\circ \pi}{180 t} \quad \dots \dots \dots \text{(VIII}_a\text{)}$$

When the *linear* velocity of any point, P , in the body, and its distance from the axis are known, the angular velocity of the body can be found. Thus :—

Let v = Component of linear velocity of P perpendicular to OP (see the previous figure).

„ r = Radius, OP .

Then, v = Arc described by P in unit time.

$$\therefore \quad \frac{v}{r} = \text{Circular measure of angle described by } OP \text{ in unit time.}$$

$$\begin{array}{l} \text{i.e.,} \\ \text{Or,} \end{array} \quad \left. \begin{array}{l} \omega = \frac{v}{r} \\ v = \omega r \end{array} \right\} \dots \dots \dots \text{(IX)*}$$

Variable Angular Velocity.—Angular Acceleration.—When the angular velocity is variable, it is measured in a way similar to that of variable linear velocity.

[Let $\Delta \theta$ = small angle described by O P, in small interval of time, Δt ; then we have :—

$$\omega = \frac{\Delta \theta}{\Delta t},$$

which, in the limit, becomes :—

$$\omega = \frac{d\theta}{dt} \dots \dots \dots \text{(X)]}$$

DEFINITION.—The angular acceleration of a rotating body is the rate of change of its angular velocity.

Angular acceleration may be either *uniform* or *variable* according as equal changes of angular velocity take place in equal or unequal intervals of time. When uniform, angular acceleration is measured by the increase or decrease of angular velocity per unit time.

Let ω_1, ω_2 = Angular velocities at the beginning and end of interval of time, t .

„ θ_1, θ_2 = Angular displacements at the beginning and end of interval of time, t .

„ α = Angular acceleration.

$$\begin{array}{l} \text{Then,} \\ \text{Or,} \end{array} \quad \left. \begin{array}{l} \alpha = \frac{\omega_2 - \omega_1}{t} \\ \omega_2 = \omega_1 + \alpha t \end{array} \right\} \dots \dots \dots \text{(XI)}$$

From these equations and those previously deduced for uniformly accelerated linear motion, the student will notice the similarity of the relations between the terms s, v , and a , and θ, ω , and α respectively.

Hence, we get the remaining and corresponding equations for rotary motion, viz. :—

$$\theta = \omega_1 t + \frac{1}{2} \alpha t^2 \dots \dots \dots \text{(XII)}$$

$$\omega_2^2 = \omega_1^2 + 2 \alpha \theta \dots \dots \dots \text{(XIII)}$$

* It is sometimes convenient to speak about the angular velocity of a *point*, such as P in the foregoing figure. Such a phrase is not strictly correct, and when used, it should be understood to mean the angle described in unit time by the radius drawn through the point, P.

[Generally, we have:—

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \dots \dots \dots (XIV)]$$

Composition and Resolution of Velocities.—A moving body may have at any instant two or more velocities in different directions, and it then becomes an important problem to be able to determine the resultant velocity, both in magnitude and in direction. Thus, the magnitude and direction of the motion of a man who walks across the deck of a moving ship is different from that of the ship and also from that of his motion relative to the deck. Similarly, the motion of a point on the rim of a carriage wheel in motion is, in general, different in magnitude and direction from its circular motion about the axle, and also from the onward motion of the wheel as a whole.

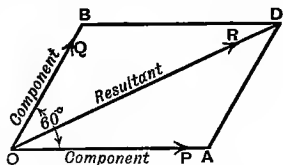
The process of finding a single velocity equivalent in effect to two or more velocities is called the *Composition of Velocities*.

The process of finding two or more velocities equivalent in effect to a single velocity is called the *Resolution of Velocities*.

DEFINITIONS.—The single velocity which is equivalent to two or more velocities is called their *Resultant*, and these two or more velocities are called the *Components*.

Parallelogram of Velocities.—If two component velocities be represented, in magnitude and direction, by two adjacent sides, O A, O B, of a parallelogram, their resultant velocity will be represented by the diagonal, O D, through their intersection.

Thus, if a moving point, O, possess simultaneously two velocities, P and Q, in directions O A and O B respectively, and, if O A and O B represent the magnitudes of these velocities, their resultant velocity, R, will be represented both in magnitude and in direction by the diagonal, O D, of the parallelogram constructed on O A, and O B, as adjacent sides.



PARALLELOGRAM LAW.

Let θ = angle between the directions of the velocities, P and Q.

„ α = $\angle A O D$, and β = $\angle B O D$, the angles between the direction of the resultant, R, and the components P and Q respectively.

Then the student may easily prove from Euclid II., 13 and 14, or by trigonometry, that:—

$$R^2 = P^2 + Q^2 + 2 P Q \cos \theta. \quad \dots (XV)$$

$$\left. \begin{array}{l} \text{And,} \quad \tan \alpha = \frac{Q \sin \theta}{P + Q \cos \theta} \\ \text{Or,} \quad \tan \beta = \frac{P \sin \theta}{Q + P \cos \theta} \end{array} \right\} \dots (XVI)$$

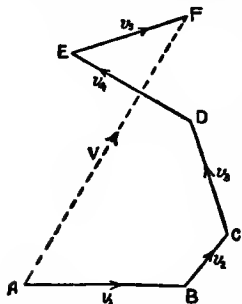
From these equations the magnitude and direction of the resultant velocity can be calculated.

It is not necessary to complete the parallelogram as explained above, it being quite sufficient to draw but one-half of the figure. Thus, A D is equal and parallel to O B; hence, as much can be determined from the triangle, O A D, as from the complete parallelogram, O A D B.

Triangle of Velocities.—If two component velocities be represented in magnitude and direction by two sides of a triangle taken in order, their resultant will be represented in magnitude and direction by the third side taken in the reverse direction.

Hence, if there be simultaneously impressed on a point three velocities represented in magnitude and direction by the sides of a triangle taken in order, then the point will remain at rest.*

Polygon of Velocities.—If several component velocities be represented by all but one of the sides of a polygon, A B C D E F, taken in order—the resultant velocity will be represented in magnitude and direction by the remaining side, A F, taken in the opposite direction.



POLYGON OF VELOCITIES.

Thus, if a moving point have simultaneously impressed upon it velocities, $v_1, v_2, \dots v_5$, and these are represented in magnitude and direction by the sides A B, B C, \dots E F of a polygon, A B C D E F, then the resultant velocity will be represented in magnitude and direction by the side, A F, required to complete the polygon.

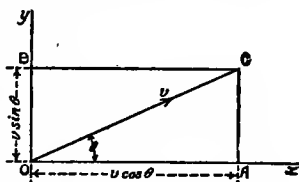
* In setting out the *Parallelogram*, or *Triangle of Velocities*, it is not necessary to draw the sides parallel to the velocities represented. The sides may be drawn in directions *perpendicular* to the respective velocities, or, indeed, at any other angle, so long as the angle is the same for all the sides. In such cases the line representing the resultant will be equally inclined to its true direction. These several proofs are applicable, in the same way, to the *Parallelogram*, *Triangle*, and *Polygon of Static Forces*, as explained in my *Elementary Manual on Applied Mechanics*.

If the figure whose sides represent the component velocities be closed or completed when the last velocity has been represented, then there is no resultant velocity, and the point will remain at rest.

It is equally important to be able to *resolve* a given velocity into two or more component velocities. Thus, the velocity, R (see the figure for *Parallelogram of Velocities*), can be resolved into two components, P and Q , in the directions OA , OB respectively. Or, the velocity, V (in the last figure), may be resolved into a number of components, v_1, v_2, \dots , in directions AB, BC, \dots . Further, the directions of the component velocities may be anything we like. Thus, in resolving a given velocity, R , into two components, we can do so in an infinite number of ways, since an infinite number of parallelograms, such as $OADB$, can be found having OD for one of their diagonals. When, however, the *directions* of the components are fixed, their magnitudes will be definite and easily determined. Referring to the figure for the *Parallelogram of Velocities*, let OD represent a velocity, R , which has to be resolved into two components in the directions OA and OB . From D draw DA parallel to BO and DB parallel to AO , meeting the lines OA and OB in the points A and B respectively. Then OA and OB represent the component velocities P and Q to the same scale that OD represents the velocity R .

The most important case of resolution is that wherein the given velocity has to be resolved into components whose directions are at right angles to each other. Thus, let it be required to resolve the velocity, v , whose direction is OC , into its **Rectangular Components** along Ox and Oy .

From C drop the perpendiculars CA , CB on the axes Ox and Oy . Then, OA , OB are the components in the required directions.



RECTANGULAR RESOLUTION.

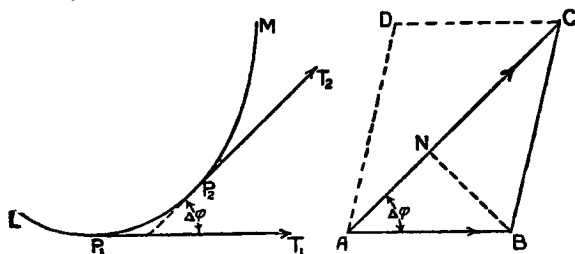
Let v_x, v_y = Components of v in directions Ox, Oy respectively.
 „ θ = Angle between the directions of v and v_x .

Then, $v_x = v \cos \theta$
 And, $v_y = v \sin \theta$ } (XVII)

Composition and Resolution of Accelerations.—Since an *acceleration* is a *rate of change of velocity*, whether in *magnitude* or in *direction*, it follows that accelerations may be compounded or resolved according to the same rules as velocities.

If the direction of motion of a body be constant, then change of velocity can only take place in that direction. Thus, if a body is constrained to move in a rectilinear path its only acceleration is one of magnitude, and takes place along the straight line in which the body moves.

Again, the velocity of a body may be constant in *magnitude*, but variable in *direction*, as in the case of a body moving with uniform speed in a circle. Or, it may vary both in magnitude and in direction, as in the case of the bob of a pendulum swinging to and fro. The Total Acceleration, in any case, may be found in the following manner :—



TOTAL ACCELERATION OF A MOVING BODY.

Let LM be the path of a moving body, and P_1 , P_2 its positions at the beginning and end of an interval of time, t .

At P_1 , its velocity is in the direction of the tangent, $P_1 T_1$, and at P_2 , its velocity is in the direction, $P_2 T_2$.

From A draw AB and AC to represent in magnitude and direction the velocities of the body at the points P_1 and P_2 respectively. Join BC, and complete the parallelogram, ABCD. Then AC represents the resultant velocity whose components are AB and AD or BC. But, if the velocity of the body had remained constant in magnitude and in direction during the time, t , its velocity at the end of that interval of time would have been represented by AB. Hence, in the above case, AD, or BC, represents, in magnitude and direction, the *change of velocity* during the time, t .

$$\therefore \quad \text{Total acceleration} = \frac{BC}{t}.$$

[Suppose the arc $P_1 P_2$ to be very small; and

Let v = Velocity of body at point, P_1 .

„ $v + \Delta v$ = Velocity of body at point, P_2 .

„ Δt = Small interval of time required to traverse the small arc, $P_1 P_2$.

„ $\Delta \phi$ = Angle between tangents to curve at P_1 and P_2 .

From B draw BN perpendicular to AC. Then BN and NC represent respectively the components of the total acceleration, BC, along lines normal and tangential to the curve at a point near to P₁.

Hence, $\text{Normal Acceleration} = \text{limit of } \frac{BN}{\Delta t}$

And, $\text{Tangential Acceleration} = \text{limit of } \frac{NC}{\Delta t}$.

Now, $BN = AB \sin \Delta \phi = v \sin \Delta \phi$.

$\therefore \text{Limit of } \frac{BN}{\Delta t} = v \frac{d\phi}{dt}$.

In the limit, let ds denote the infinitesimally small arc, P₁P₂, and let ρ denote the radius of curvature at the point, P₁ or P₂.

Then, $v \frac{d\phi}{dt} = v \frac{ds}{dt} \cdot \frac{d\phi}{ds} = v^2 \frac{d\phi}{ds}$.

But, from the properties of plane curves, we know that:—

$$\frac{d\phi}{ds} = \frac{1}{\rho}.*$$

$\therefore \text{Normal Acceleration} = \frac{v^2}{\rho} \dots \dots \text{(XVIII)}$

Again, $\text{Limit of } \frac{NC}{\Delta t} = \text{limit of } \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$.

$\therefore \text{Tangential Acceleration} = \frac{dv}{dt} \dots \dots \text{(XIX)}$

The result expressed in equation (XIX) agrees with the corresponding general equation previously deduced.

In the case of a body moving in a circle with uniform motion, we get $\rho = r = \text{radius of circle}$, and v is constant. Then the tangential acceleration is *nil*, and the

$\text{Normal or Radial Acceleration} = \frac{v^2}{r} \dots \text{(XVIII}_a\text{)}$

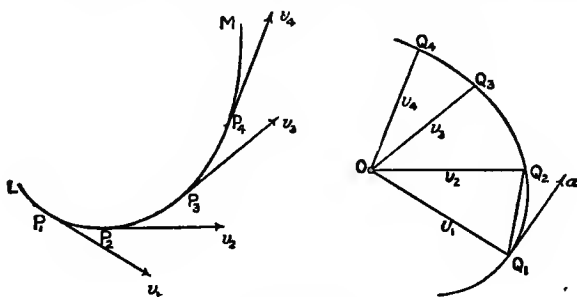
This is usually spoken of as the *Centripetal Acceleration*.]

The Hodograph—Uniform Motion in a Circle.—We shall now extend the foregoing principles to the determination of the acceleration of a body which moves with uniform velocity in a circle. In the first place we shall briefly describe the properties of the *Hodograph*.

* See Todhunter's *Dif. Calculus*, p. 343.

DEFINITION.—If a point, P, be moving in any manner in a straight or curved path, and if from a fixed point, straight lines be drawn representing in magnitude and direction the velocities of P at different points of its path, the locus of the extremities of those lines will be a curve which is the Hodograph of P's motion.

Thus, let LM be the path of a moving point, P. Let the velocities at the points $P_1, P_2, P_3 \dots$ be $v_1, v_2, v_3 \dots$. From any point, O, draw $OQ_1, OQ_2, OQ_3 \dots$ respectively parallel to $v_1, v_2, v_3 \dots$ and of lengths representing these velocities. Then the curve, Q_1, Q_2, Q_3, Q_4 , which is the *locus* of



THE HODOGRAPH.

the point Q_i is the hodograph of P's motion in the path, LM. Hence, to every point on the curve, LM, there will be a corresponding point on the hodograph, so that while the body describes the curve, LM, we may imagine a point to describe the hodograph.

We shall now prove the following property of the hodograph:—

The acceleration of the body at any point on the curve, LM, is represented in magnitude and direction by the velocity of the corresponding point on the hodograph.

Let $v =$ Average velocity between P_1 and P_2 .

„ $\Delta t =$ Indefinitely small time required to describe arc $P_1 P_2$.

Then,
$$v = \frac{P_1 P_2}{\Delta t}.$$

But OQ_1, OQ_2 , represent the velocities of the body at the beginning and end of the interval of time, Δt . Therefore, chord $Q_1 Q_2$ represents the *change of velocity* of the body, during that interval of time.

That is,
$$\left. \begin{array}{l} \text{Acceleration of body between} \\ P_1 \text{ and } P_2 \end{array} \right\} = \frac{Q_1 Q_2}{\Delta t}.$$

But, in the limit, when P_2 approaches indefinitely near to P_1 , and, therefore, also Q_2 approaches indefinitely near to Q_1 , we get:—

$$\text{Chord } Q_1 Q_2 = \text{Arc } Q_1 Q_2.$$

$$\text{But, } \frac{\text{Arc } Q_1 Q_2}{\Delta t} = \text{Velocity of } Q \text{ in hodograph.}$$

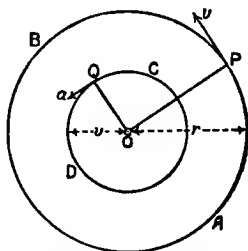
$$\therefore \left. \begin{array}{l} \text{Acceleration of} \\ \text{body in curve} \\ \text{LM} \end{array} \right\} = \text{Velocity of } Q \text{ in hodograph.}$$

Again, since the direction of motion of a point on a curve is along the tangent to the curve at that point, so the direction of motion of Q on the hodograph at any point is along the tangent to the hodograph at that point. Hence, the *direction* of the acceleration of the moving body at any point on the curve, LM, is represented by the tangent at the corresponding point on the hodograph.

Thus, let P_1 and Q_1 be corresponding points on the path and hodograph respectively. Then, OQ_1 represents the velocity of the body at P_1 , and the tangent to the hodograph at Q_1 represents the *direction* of the acceleration at the same point.

When a body describes a circle with uniform velocity, it is evident that there can be no tangential acceleration.

Let APB represent the circular path of a body moving with uniform velocity, v . Then, it is clear that the hodograph of the moving body will also be a circle whose radius is v . With centre, O, and radius representing v , describe a circle, CQD. Then, circle CQD is the hodograph. Let P be the position of the body at any instant. Draw the radius, OQ, of the hodograph parallel to the tangent at P; or, what is the same thing, draw OQ perpendicular to OP. Since the radius, OP, describes equal angles in equal times, it follows at once that the radius, OQ, of the hodograph will also describe equal angles in equal times. In other words, the velocity of Q in the hodograph is uniform. Now, the magnitude and direction of the *velocity* of Q represent the magnitude and direction of the *acceleration* of P. Therefore, the direction of the acceleration of P is that of the tangent to the hodograph at the point, Q; that is, it



HODOGRAPH FOR UNIFORM MOTION IN A CIRCLE.

is along the radius, PO . As to the magnitude of this acceleration we observe that:—

Acceleration of $P = a = \text{Velocity of } Q$.

Since Q describes the circle, CQD , in the same time, t , as P describes the circle, APB , we get:—

$$\text{For hodograph, } t = \frac{\text{Circumference of } CQD}{\text{Velocity of } Q} = \frac{2\pi \times v}{a} \quad (1)$$

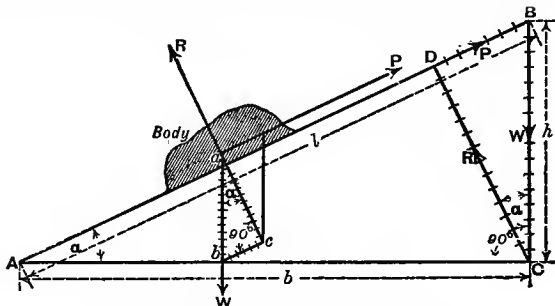
$$\text{For Path of } P, \quad t = \frac{\text{Circumference of } APB}{\text{Velocity of } P} = \frac{2\pi \times r}{v} \quad (2)$$

Equating (1) and (2) we get:—

$$\frac{2\pi v}{a} = \frac{2\pi r}{v}.$$

$$\therefore \text{Acceleration of } P = a = \frac{v^2}{r}. \quad \dots \dots \dots (XX)$$

EXAMPLE I.—A body slides down a smooth inclined plane, determine its velocity at the foot of the plane. If the plane has



MOTION ON SMOOTH INCLINED PLANE.

a rise of 25 per cent., what distance would a body, descending along it from a state of rest, describe in five seconds? Find also the time occupied in sliding down the first 50 feet of the length of the plane.

ANSWER.—Let $a = \text{Acceleration of the body along the plane.}$

„ $\alpha = \text{Inclination of plane to the horizon.}$

(1) If the body were free to move vertically downwards its acceleration in that direction would be g . But since it is con-

strained to move in the direction B A, its acceleration in this direction will be less, being, in fact, the component of g , along B A.

Hence, resolve g into its rectangular components in directions along and at right angles to B A. Then :—

$$\text{Acceleration along B A} = g \sin \alpha.$$

$$\text{Acceleration perpendicular to B A} = g \cos \alpha.$$

The latter component has no effect on the motion of the body. Hence :—

$$\text{Acceleration down the plane} = a = g \sin \alpha. \quad (1)$$

Let t = Time required to slide along a length, s .

„ v = Velocity at the end of time, t .

Then, from equation (IV_a) $v = a t$.

$$\therefore v = g t \sin \alpha. \quad (2)$$

$$\text{From equation (V_a) } s = \frac{1}{2} a t^2.$$

$$\therefore s = \frac{1}{2} g t^2 \sin \alpha. \quad (3)$$

$$\text{And from equation (VI_a) } v^2 = 2 a s.$$

$$\therefore v^2 = 2 g s \sin \alpha.$$

$$\text{But, } s \sin \alpha = \text{Height of plane of length, } s, \\ , = h, \text{ say.}$$

$$\text{Then, } v^2 = 2 g h. \quad (4)$$

That is,—*The velocity acquired by a body in sliding down a smooth inclined plane is the same as that acquired by a body falling freely through a distance equal to the height of the plane.*

From the given data, we get :—

$$\sin \alpha = \frac{25}{100} = .25,$$

$$t = 5 \text{ seconds.}$$

∴ From equation (3), we get :—

$$s = \frac{1}{2} g t^2 \sin \alpha = \frac{1}{2} \times 32.2 \times 5 \times 5 \times .25$$

$$,, = 100.625 \text{ feet.}$$

(2) Here, $s = 50$ feet, and we require t .

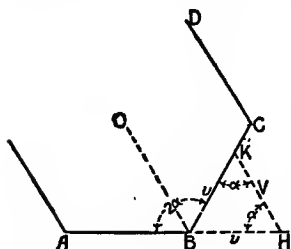
From equation (3), we get :—

$$s = \frac{1}{2} g t^2 \sin \alpha.$$

$$\therefore t = \sqrt{\frac{2s}{g \sin \alpha}} = \sqrt{\frac{2 \times 50}{32.2 \times .25}} = 3.52 \text{ seconds.}$$

EXAMPLE II.—State the rule for the composition of two velocities. If a particle describes the perimeter of a regular polygon with a constant velocity, v , show that there must be impressed on it, at each angular point a velocity equal to $2v \cos \alpha$, directed towards the centre of the circumscribing circle, where α denotes half an angle of the polygon. (S. & A. Hons. Theor. Mechs. Exam., 1885.)

ANSWER.—For answer to first part, see *Parallelogram of Velocities*. Let $A B C D \dots$ represent the sides of a regular polygon, whose centre is O . When the particle arrives at B , its direction of motion is suddenly changed from $A B$ to $B C$, while the magnitude of the velocity remains unaltered. To find the magnitude and direction of the velocity which must have been imparted to the particle at the point, B , we may proceed as follows:—



TO ILLUSTRATE EXAMPLE IV.

Produce $A B$, and set off $B H$, to represent the velocity, v , of the particle along $A B$, and $B K$ along $B C$, to represent the velocity in that direction. Then $H K$ represents in magnitude and direction the change of velocity which must have been imparted to the particle at the point, B . The magnitude of this velocity can be found from the triangle, $B H K$, or equation (XV).

$$\text{For, } V^2 = v^2 + v^2 - 2v^2 \cos (180^\circ - 2\alpha) = 2v^2 (1 + \cos 2\alpha)$$

$$,, = 4v^2 \cos^2 \alpha. \quad [\text{Since } 1 + \cos 2\alpha = 2\cos^2 \alpha.]$$

$$\therefore V = 2v \cos \alpha.$$

Join B to O , the centre of the polygon, and we get:—

In triangle $B H K$;

$$\text{Exterior } \angle A B C = \angle B H K + \angle B K H.$$

$$\text{But, } \angle B H K = \angle B K H,$$

$$\therefore \angle B H K \text{ or } \angle B K H = \frac{1}{2} \angle A B C = \alpha.$$

$$\therefore H K \text{ is parallel to } B O, \text{ since } B O \text{ bisects } \angle A B C.$$

Therefore, the velocity impressed on the particle at B is directed along $B O$, towards the centre.

EXAMPLE III.—Find, at any instant, the magnitude and direction of the velocity of a point on the rim of a wheel which rolls along a road with a constant speed, v .

The point, A, is called the Instantaneous Centre* of motion for all points on the rim of the wheel; because any point, such as P, is moving at any instant on the circumference of a circle having A for its centre and AP as its radius.

The direction of the actual motion of any point, P, is, at any instant, inclined to the horizontal at an angle equal to $90^\circ - \frac{\theta}{2}$.

EXAMPLE IV.—In the previous example find the magnitude and direction of the actual velocity of the point, P, when the radius, OP, makes angles of 0° , 90° , 180° , and 270° with the vertical radius, OA. Also, find the position of P when the resultant velocity, V, is equal to v .

ANSWER.—(1) When $\theta = 0^\circ$. From equation (1), Example III., we get :—

$$V = 2v \sin \frac{\theta}{2} = 0, \text{ since } \sin 0^\circ = 0.$$

i.e., the point is at rest when it is in contact with the ground A.

$$(2) \text{ When } \theta = 90^\circ. \text{ Here } \sin \frac{\theta}{2} = \sin 45^\circ = \frac{\sqrt{2}}{2}.$$

$$\therefore V = 2v \sin \frac{\theta}{2} = 2v \times \frac{\sqrt{2}}{2} = v\sqrt{2}.$$

$$\text{Also, Direction which } \left. \begin{array}{l} V \text{ makes with} \\ \text{the horizon} \end{array} \right\} = 90^\circ - \frac{\theta}{2} = 45^\circ.$$

$$(3) \text{ When } \theta = 180^\circ. \sin \frac{\theta}{2} = \sin 90^\circ = 1.$$

$$\therefore V = 2v \sin \frac{\theta}{2} = 2v \times 1 = 2v.$$

$$\text{And, Direction which } \left. \begin{array}{l} V \text{ makes with} \\ \text{the horizon} \end{array} \right\} = 90^\circ - \frac{\theta}{2} = 0^\circ.$$

That is, when P is vertically over O, it is moving horizontally with a velocity equal to twice the speed of the wheel.

$$(4) \text{ When } \theta = 270^\circ. \sin \frac{\theta}{2} = \sin 135^\circ = \frac{\sqrt{2}}{2}.$$

$$\therefore V = 2v \sin \frac{\theta}{2} = 2v \times \frac{\sqrt{2}}{2} = v\sqrt{2}.$$

$$\text{And, The inclination } \left. \begin{array}{l} \text{of } V \text{ to the} \\ \text{horizon} \end{array} \right\} = 90^\circ - 135^\circ = -45^\circ.$$

* The Instantaneous Centre of Motion is that point about which, for the instant, we may regard any other part of the body as rotating.

(5) To find θ when $V = v$.

Here,
$$V = 2v \sin \frac{\theta}{2}.$$

$$\therefore \sin \frac{\theta}{2} = \frac{V}{2v} = \frac{1}{2}.$$

$$\therefore \frac{\theta}{2} = 30^\circ, \text{ or } 150^\circ,$$

$$\therefore \theta = 60^\circ, \text{ or } 300^\circ.$$

These agree with the two positions of P when the chord, AP, is equal to the radius, OA.

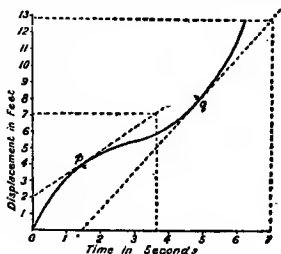
Varying Velocities.—When a body is caused to move through different distances in equal times, its speed or velocity is said to be variable. The velocity of the body at a particular instant, or during a very short interval of time, such as the fraction of a second, will be the distance it travels during that time.

The instantaneous velocity of a body moving with a variable speed may be graphically represented and estimated in the following manner:—Plot an abscissa

or horizontal line to represent seconds and fractions of a second, and an ordinate or vertical line to represent displacement in feet. Then, suppose that the curved line represents to scale the varying velocity of a body, and we desire to know its instantaneous values at the positions *p* and *q*.

(1) Draw a tangent to the curve at the point *p*. Then, from the slope of this tangential dotted line we see, that if the velocity of the body had been uniform whilst passing through a point represented by *p*, a distance of $(7 - 2) = 5$ feet, would have been passed over in 3.75 seconds. Consequently, we estimate that its *instantaneous velocity* at *p* was $(5 \div 3.75) = 1.3$ feet per second. In the same way, we find that had the velocity of the body remained uniform whilst passing through a point represented by *q* on the curve, then 13 feet would have been passed over in $(7 - 1.5)$ or 5.5 seconds. Therefore its *instantaneous velocity* at *q* $= (13 \div 5.5) = 2.36$ feet per second.

In this way the velocity of the body may be ascertained at any other instant.



VARIABLE VELOCITIES.

LECTURE X.—QUESTIONS.

1. Define the terms velocity and acceleration, and explain how these are measured—(1) when uniform; (2) when variable. Give examples of bodies having accelerations—(a) constant in magnitude and direction; (b) constant in magnitude but not in direction; (c) constant in direction but not in magnitude; (d) variable both in magnitude and direction.

2. When the velocity of a particle is uniformly accelerated, show that $s = \frac{1}{2} a t^2$. A particle moves from a state of rest under the action of a force which increases its velocity by 20 feet per second in every second of its motion. After four seconds the force ceases to act on it. What distance does it describe in the first six seconds of its motion? (S. & A. Adv. Theor. Mechs. Exam., 1880.) *Ans.* 320 feet.

3. Establish the formulæ for uniform acceleration in the direction of motion:— $v_2 = v_1 + a t$; $s = v_1 t + \frac{1}{2} a t^2$, and from these results deduce the formula— $v_2^2 = v_1^2 + 2 a s$. Find an expression for the distance described in the n th second.

4. A cage is ascending the shaft of a mine at a uniform rate of 10 feet per second. When it is 50 feet from the top the speed is diminished, so that it now moves with a uniformly retarded velocity, and finally comes to rest at the top. Find the retardation. *Ans.* 1 foot per second per second.

5. State the rule for the composition of two velocities. Draw two lines, A B, A C, containing an acute angle. A particle is at A moving with a given velocity, V, from A towards B. Give a construction for determining the velocity that must be impressed on it, to make it move with a velocity, 2 V, from A towards C.

6. A particle describes the perimeter of a regular hexagon with a constant velocity of 100 feet a second. Find the magnitude and direction of the velocity that must be communicated to it, at the instant it reaches an angular point. *Ans.* 100 feet per second towards centre of hexagon.

7. Two bodies start together from rest, and move in directions at right angles to each other. One moves with a uniform velocity of 3 feet per second, while the motion of the other is uniformly accelerated. At the end of four seconds the bodies are found to be 20 feet apart. Determine the acceleration of the latter body. *Ans.* 2 feet per second per second.

8. Two bodies, P and Q, move with different velocities along the same line. What is the relative velocity of Q to P? If Q is allowed to fall freely, and two seconds after P is allowed to fall freely from the same point, find the relative velocity of Q to P at any subsequent time.

Ans. 64.4 feet per second.

9. Define angular velocity. P is a point of a body turning uniformly round a fixed axis, and P N is a line drawn from P at right angles to the axis. If P N describes an angle of 375° in three seconds, what is the angular velocity of the body? and if P N is 6 feet long, what is the linear velocity of P? *Ans.* (1) 0.7π radians per second; (2) 4.2π feet per second.

10. A point is describing a circle of radius 21 feet, with a uniform velocity of 12 feet per second. Find the change in its velocity after it has described one-sixth of a whole circumference. *Ans.* 12 feet per second, at 120° with first direction.

11. A wheel, whose diameter is 5 feet, turns forty times a minute; find its angular velocity and the linear velocity of a point on its circumference. If the centre of the wheel moves in a straight line with a velocity of 20 miles an hour; what are the velocities, relative to a very distant fixed point in the straight line, of the ends of the diameter which is at any instant vertical? *Ans.* (1) $\omega = \frac{4\pi}{3}$ radians per second; (2) 10.5 feet per second; (3) upper end = 40 miles per hour; lower end = zero.

12. What is the numerical value of the angular velocity of a body which turns uniformly round a fixed axis twenty-five times a minute? A B C is a triangle right angled at C. It is turning with a given angular velocity, ω , round an axis through A, at right angles to its plane. Find the magnitude and direction of the velocities of B and C; and also the relative velocity of B to C. *Ans.* $\frac{5}{6}\pi$ radians per second.

13. A train descending a gradient increases its speed from 40 to 49 miles per hour in four and a-half minutes. Find the mean acceleration. Taking the acceleration due to gravity at 32 in feet and seconds, determine the gradient. *Ans.* (1) 0.049 foot per second per second, or 120 miles per hour per hour; (2) 1 in 654.

14. Given the base, b , of a smooth inclined plane, find its height, h , so that the horizontal component of the velocity of a body at the foot of the plane shall be a maximum. *Ans.* $h = b$.

15. Define the hodograph, and prove that the acceleration of a point's motion is equal to the velocity with which the hodograph is traced out. Determine, by means of the hodograph, the acceleration of a body which moves with uniform velocity in a circle.

16. Define the angular velocity of a moving point with respect to a fixed point. Under what circumstances will the angular velocity of the moving point be equal to its linear velocity divided by its distance? Draw an equilateral triangle A B C, having each side 12 feet long; a point moves along B C with a velocity of 10 feet a second; when it is at C, what is its angular velocity with respect to A?

Ans.

17. Two circles touch each other externally, and the point of contact (A) is in the same vertical line as the centres; from any point (P) of the upper circumference draw a straight line P A Q to meet the lower circumference in Q; if a particle is allowed to fall from P along P Q, show that the time it takes to reach Q is constant for all positions of P. Also compare the times in which P A and A Q are described.

Ans. $\sqrt{\frac{2R}{g}}$

18. A body weighing 322 lbs. is lifted by a force of F lbs. which alters. When the body has risen through the distance x feet, the force in lbs. for the several values of x is as follows (or would be if the body rose as far):—

x	0	1	2	3	4	5.5	7	9	11	12.5	14	17	20
F	540	540	540	530	500	460	310	220	190	190	190	190	190

Using squared paper, find the velocity in each position and the time taken by the body to get to each position counting from $x = 0$, the velocity then being 5 feet per second.

19. A body weighing 3,220 lbs. is lifted vertically by a rope, there being a damped spring balance to indicate the pulling force F pounds of the rope. When the body has been lifted x feet from its position of rest the pulling force is automatically recorded as follows:—

x	0	18	43	60	74	95	111	130
F	7,700	7,680	7,430	7,130	6,770	5,960	5,160	3,970

Using squared paper, find the velocity, v , for values of x of 20, 50, 80, 120, and draw a curve showing the probable values of v for all values of x . In what time does the body get from $x = 75$ to $x = 85$? *Ans.* 34·6; 53·6; 65; 69·5 feet per second; 0·154 second. *Ans.* 740,600 ft.-lbs.; 370,300 ft.-lbs.; 370,300 ft.-lbs. (B. of E., Adv. & H., Part I., 1900.)

N.B.—See Appendices B and C for other questions and answers.

LECTURE X.—A M. INST. C. E. EXAM. QUESTIONS.

1. Having given a diagram setting out the relation between distances and times in the movement of a body, show how you might use it to find the velocity at any time. What form does such a diagram take in the case of a body starting from rest with a uniform acceleration? (I.C.E., Oct., 1897.)

2. In plane motion of a rigid body, given the acceleration of a point A and the path of a point B, show how we find a diagram for the acceleration of any point in the body. Prove your construction to be correct.

(I.C.E., Oct., 1898.)

3. Prove that if the link and force polygons are both closed, a system of forces in a plane is in equilibrium. (I.C.E., Oct., 1898.)

4. Two bodies, A and B, are moving along any two straight lines; state what is meant by the velocity of A relatively to B, and explain how you would determine this having given the velocities of A and B. A train is travelling at the rate of 20 miles an hour, and a man, sitting in a corner of a compartment with both windows down, observes a stone pass in a straight line at right angles to the length of the train through both windows. If it appears to the man to have a velocity of 20 feet per second, with what horizontal velocity must the stone have been thrown?

(I.C.E., Feb., 1899.)

5. Draw a rectangle ABCD having AB = 4 inches and BC = 2 inches. A body is acted on by a force P of 142 tons weight in the direction from C to B. You are required to find, graphically, three forces which will balance P, their lines of action being given as follows:—i. A line through D and a point E which is on AB, and 1 inch from A. ii. A line through B at right angles to DE. iii. The line CD. Show clearly, by means of arrows, the direction of each force. (I.C.E., Feb., 1899.)

6. What is meant by the instantaneous centre of a piece moving in a fixed plane? A rigid body has a plane motion. Three points on the body A, B, and C, in the same plane, are such that AB = 3 feet, BC = 2 feet, and AC = 2.6 feet. At a certain instant it is known that the point A has a velocity of 4 feet per second in the direction from A to C, and that the point B is moving in the direction from C to B. Show how the velocity of any other point on the body may be obtained, graphically or otherwise, and determine the values for the velocities of B and the point midway between A and B. (I.C.E., Feb., 1899.)

7. A body of 250 lbs. is acted on by a resultant force, F , which varies in amount. When the body was passed through the distance x feet, the force in lbs. weight is as follows:—

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
F	25	26	26	24	21.5	20	17	12.5	7	3	0

Find, either by a graphical or arithmetical method—(1) The average force acting on the body during the first foot. (2) The work done on the body during the first half of a foot. (3) If the velocity is 4 feet per second when $x=0$, find what it is when $x=0.5$ foot. (I.C.E., Feb., 1899.)

8. Show that the parallelogram of force applies also to couples, and that the parallelogram of velocities is applicable to rotations.

(I.C.E., Oct., 1899.)

9. Draw a curve of velocities referred to time which might apply to the starting of a train from rest, and show how to deduce the curve of acceleration from it. (I.C.E., Oct., 1899.)

10. A body moving at a velocity of 10 feet per second to start with is acted upon by forces which produce an acceleration varying uniformly from 10 foot-seconds per second to 20 foot-seconds per second during the 10 seconds they act; find the distance the body travels during the time.

(I.C.E., Oct., 1899.)

11. What do you understand by the instantaneous centre of motion? Four links are pivoted together in a plane; find the instantaneous centres of motion of the opposite links with respect to each other, with proof of same. (I.C.E., Oct., 1899.)

12. A body is rotating about an axis with an angular velocity of 4 radians per second, and about an axis intersecting the former at an angle of 60° with an angular velocity of 9 radians per second; find the axis about which the resultant rotation is taking place and its amount.

(I.C.E., Feb., 1900.)

13. Define acceleration. Find an expression for the acceleration of a weight, W , rotating at a uniform angular velocity round an axis at the end of an arm whose weight may be neglected. (I.C.E., Feb., 1900.)

14. A vehicle is proceeding in a westerly direction at the rate of 10 miles per hour. The wind is blowing from the south-west at the rate of 5 miles per hour. Find the velocity of the wind relatively to the vehicle in amount and direction. If the vehicle returns at the same rate, what is the velocity and direction of the wind relatively to it? (I.C.E., Feb., 1900.)

15. What do you mean by an instantaneous centre of rotation? Give an example. What points on the rim of a carriage wheel are moving at the same rate as the middle of the axle? (I.C.E., Feb., 1901.)

16. Explain the usual British and Continental units of mass, force, energy, power, and heat. Given that 1 kilogram equals 2.2 lbs., compare the velocities generated in 1 second by (1) the weight of 1 oz. acting on a kilogram, and (2) the weight of a gram acting on 1 lb. (I.C.E., Feb., 1901.)

17. Give a proof of the formulæ for falling bodies, (a) $v = gt$, (b) $s = \frac{1}{2}gt^2$, (c) $v^2 = 2gs$, where g is the acceleration due to gravity. Find the distance traversed by a falling body starting from rest, during the 8th and 10th seconds of its motion. (I.C.E., Oct., 1901.)

18. State the proposition of the parallelogram of forces and describe an experiment to verify it. Find by calculation the magnitude of the resultant of two forces of 5 lbs. and 10 lbs. weight acting at an angle of 60° . (I.C.E., Oct., 1901.)

19. A cam moves a roller up and down between vertical guides, the displacement of the roller being noted for each twelfth of a revolution of the cam as follows:—

Twelfths of revolution of cam.	1	2	3	4	5	6	7	8	9	10	11	12
Displacement of roller in inches.	0.19	0.97	2.15	4	5.41	6	5.4	4.01	2.16	0.97	0.24	0.00

Plot on squared paper to a suitable scale the displacement or space curve of the roller, and from it determine points in and draw the curves representing the velocity and acceleration of the roller for a complete revolution of the cam. Assuming the angular velocity of the cam shaft to

be uniform, and the time of one revolution to be $\frac{1}{2}$ second, find from the curves the numerical value of the maximum velocity and acceleration of the cam roller. (I.C.E., Oct., 1901.)

20. Prove that the velocity acquired by a body in sliding down a smooth inclined plane is the same as that acquired in falling freely through a distance equal to the height of the plane. (I.C.E., Feb., 1902.)

21. State the principle of the "parallelogram of velocities." If a man rows a boat at the rate of 3 miles an hour in a direction 60° east of north, and in a current which flows due south at the rate of $1\frac{1}{2}$ miles an hour, find the direction the boat will move and the rate at which it will move.

(I.C.E., Feb., 1902.)

22. Explain how to determine graphically the relative velocity of two points, the magnitudes and directions of whose velocities are known. Find the true course and velocity of a steamer steering due north by compass at 12 knots through a 4-knot current setting south-west, and determine the alteration of direction by compass in order that the steamer should make a true northerly course. (I.C.E., Oct., 1902.)

23. Explain what is meant by the instantaneous axis of a link. A horizontal engine has a connecting-rod five cranks in length. Find the velocity ratio of slide-block and crank-pin at one quarter stroke. What is the mean velocity ratio of crank-pin and slide-block? (I.C.E., Feb., 1903.)

24. Explain how to determine the relative velocity of two bodies. A is travelling due north at constant speed. When B is due west of A and at a distance of 21 miles from it, B starts travelling north-east with the same constant speed as A. Determine graphically, or otherwise, the least distance which B attains from A. (I.C.E., Feb., 1903.)

N.B.—See Appendices B and C for other questions and answers.

LECTURE XI.

CONTENTS.—Quantity of Motion—Definition of Momentum—Example I.
 Newton's Laws of Motion—Examples II. and III.—Motion on a Double
 Inclined Plane—Examples IV. and V.—Energy—Definition of Energy
 —Definitions of Potential and Kinetic Energy—Expression for Kinetic
 Energy—Energy Equations—Examples VI., VII., and VIII.—
 Questions.

Quantity of Motion.—In the preceding Lecture we have confined our attention chiefly to cases of pure motion—that is, motion considered apart from mass and force. In this Lecture we shall treat of the motion of bodies as produced by the action of external forces, and establish the relations between the *quantity of motion* thus produced and the magnitude of the forces producing it. Quantity of motion is measured by the product of the *mass* and its *velocity*. The term **Momentum** is used instead of *Quantity of motion*, and hence we get the following:—

DEFINITION.—The momentum of a moving body is the product of its mass and velocity.

Thus, let m be the mass, and v the velocity of a body:—

Then, $\text{Momentum} = m v$.

EXAMPLE I.—Of two steam hammers, one weighs 5 tons and the other 10 tons. The former has a drop of 10 feet and the latter 6 feet. Compare their momenta at the end of their respective strokes.

ANSWER.—In order to find their velocities at the moment of impact, we may employ formula (VI_b) of Lecture X.:—

$$v^2 = 2 g s,$$

∴ for the first hammer, $v_1 = \sqrt{2 \times 32 \times 10} = 8\sqrt{10}$ ft. per sec.

And, for the second } $v_2 = \sqrt{2 \times 32 \times 6} = 8\sqrt{6}$ „
 hammer,

$$\therefore \left. \begin{array}{l} \text{Momentum of} \\ \text{first hammer} \\ \text{Momentum of} \\ \text{second hammer} \end{array} \right\} = \frac{m_1 v_1}{m_2 v_2} = \frac{5 \times 8\sqrt{10}}{10 \times 8\sqrt{6}} = \frac{\sqrt{5}}{2\sqrt{3}} = \frac{1}{1.549}.$$

Newton's Laws of Motion.—The three fundamental laws of Dynamics, called *Laws of Motion*, were first clearly set forth by Newton, and may be stated as follows:—

LAW I. (*Law of Inertia*).—Every body continues in a state of rest, or of uniform motion in a straight line, except in so far as it may be compelled to change that state by external force acting on it.

LAW II. (*Law of Force and Motion*).—Rate of change of momentum is proportional to the force which causes it, and takes place in the direction of the force.

LAW III. (*Law of Stress*).—When two bodies mutually act upon each other, the momenta developed in the same time are equal but opposite in direction.

Or, To every action there is an equal and opposite reaction.

Law I.—This Law asserts that matter is *indifferent* to motion, *i.e.*, has no *innate* tendency to start into motion when at rest, nor to change its motion, either in magnitude or in direction, when once it is made to move. Hence, a body at rest or in motion, and unacted upon by force, will continue to remain at rest, or to move on in a straight line with uniform motion. Should any change take place in its motion, then we immediately infer that the body has been acted upon by some external force. This tendency of matter to resist change in its state of rest or of uniform motion in a straight line is called *Inertia*, and the first Law is often spoken of as the *Law of Inertia*.

Law II.—The first Law asserts that change of momentum is caused by the action of force, and the second Law gives us a means of measuring this force, *viz.*, that the force is proportional to the *rate of change of momentum*.

Let F = Force producing change of momentum.

„ m = Mass of body.

„ v_1, v_2 = Initial and final velocities of body.

„ t = Time during which F acts.

Then, $\text{Change of momentum} = m(v_2 - v_1).$

And, $\left. \begin{array}{l} \text{Rate of change of} \\ \text{momentum} \end{array} \right\} = \frac{m(v_2 - v_1)}{t}.$

∴ By the *Second Law of Motion*, we get :—

$$F \propto \frac{m(v_2 - v_1)}{t}.$$

Or,
$$F = \frac{m(v_2 - v_1)}{t} \times \text{constant}.$$

It now remains to establish the exact relation between those quantities. If we accept the definition of *Unit Force* given on page 2, Lecture I., Vol. I., as being *that force which, acting for unit time on unit mass, produces unit change of velocity*, we find the numerical value of the *constant* in the above equation to be *unity*.

i.e.,
$$F = \frac{m(v_2 - v_1)}{t} \text{ (pounds)}.$$

But we have shown in Lecture X. that :—

$$a = \frac{v_2 - v_1}{t} \text{ (change of velocity per unit of time),}$$

where a denotes the acceleration produced when the motion is uniformly accelerated.

∴
$$F = m a \text{ (mass} \times \text{its acceleration).}$$

The above definition is that of the *Absolute Unit* of Force; and, therefore, the force, F , as given by these equations, is expressed in absolute units. Engineers, however, prefer measuring their forces by the weights which they are capable of supporting, and the above equations may be modified to suit these units. Let P, P_1 , be the statical measures of the forces required to produce accelerations, a, a_1 , on a given mass, m ; then by Law II., we get :—

$$P : P_1 = a : a_1.$$

If one of these forces be that due to gravity, viz., the weight, w , of a body, then the acceleration is g , and we get :—

$$P : w = a : g.$$

Or,
$$P = \frac{w a}{g} \text{ (I)}$$

This equation expresses the force, P , in the same units as w , and if w be stated in *pounds weight* that will be in what we have previously called *gravitation units*.

Law III.—This Law asserts that when two bodies mutually act upon each other, the momenta generated in each are equal, but in opposite directions. Thus, when a shot is fired from a gun, the force of the explosion produces momentum in the gun equal in amount to that of the shot, and causes the recoil. We shall, however, see later on that the other effects produced in the gun and the shot are not numerically equal. In the case of mutual action between two bodies incapable of relative motion, the Law asserts that they act and react on each other with equal forces. Thus, a weight lies on a table, and presses on it with a certain force; then the table reacts on the weight with an equal and opposite force, so that every action is accompanied by an equal and opposite reaction.

The truth of this Law has been assumed throughout the whole of the preceding parts of this treatise—viz., that the effort exerted between two bodies is always equal to the resistance overcome. The two equal and opposite forces caused by the mutual action between two bodies are together spoken of as a *Stress*, and for this reason the above Law is sometimes called the *Law of Stress*. The subject of *internal stress* will be discussed in another part of this work.

We shall now apply the preceding results to some examples.

EXAMPLE II.—A 40-lb. shot is fired from a 5-ton gun with an initial velocity of 1,500 feet per second. Find the velocity of the gun's recoil, and the mean force of the explosion, supposing the gun to be 10 feet long.

ANSWER.—Let W, w = Weight of gun and shot respectively.

„ V, v = Velocity „ „

(1) By the *Third Law* :—

Momentum of gun = momentum of shot.

$$\therefore W V = w v$$

$$\text{i.e., } 5 \times 2240 \times V = 40 \times 1500,$$

$$\therefore V = \frac{40 \times 1500}{5 \times 2240} = 5.36 \text{ ft. per sec.}$$

(2) In order to find the mean effort exerted during the explosion of the powder, we must first determine the acceleration of the shot along the muzzle of the gun. Since the gun is 10 feet long, and the velocity of the shot as it leaves the

gun is 1,500 ft. per second, we get, from the formula (VI_a), Lecture X. :—

$$v^2 = 2 a s,$$

$$\therefore 1500^2 = 2 \times a \times 10,$$

$$\therefore a = \frac{1500^2}{20} = 112,500 \text{ ft. per second per second.}$$

$$\text{But, } P = \frac{w}{g} \times a,$$

$$\therefore P = \frac{40}{32} \times 112,500 = 140,625 \text{ lbs.}$$

EXAMPLE III.—A railway train, exclusive of engine, weighs 200 tons, and moves on a level line. In 10 minutes its speed is increased from 10 miles per hour to 40 miles per hour. Determine the mean pull between the engine and train, the frictional resistances being taken at 10 lbs. per ton.

ANSWER.—The pull between the engine and train consists of two parts; (1) the force required to accelerate the train, and (2) the force required to overcome the frictional resistances.

(1) Let P_1 = Force required to accelerate the train,

$$\text{Then, } P_1 = \frac{w a}{g} = \frac{w (v_2 - v_1)}{g t}.$$

$$\text{But, } v_1 = 10 \text{ miles per hour} = \frac{44}{3} \text{ ft. per second.}$$

$$v_2 = 40 \quad \quad \quad = \frac{176}{3} \quad \quad \quad "$$

$$\text{And, } t = 10 \text{ minutes} = 600 \text{ seconds.}$$

$$\therefore P_1 = \frac{200 \times 2240 \times \left(\frac{176}{3} - \frac{44}{3} \right)}{32 \times 600} = 1026.6 \text{ lbs.}$$

(2) The resistance of friction being 10 lbs. per ton,

$$\text{The total frictional resistance} = P_2 = 200 \times 10 = 2000 \text{ lbs.}$$

$$\therefore \left. \begin{array}{l} \text{Mean pull between} \\ \text{engine and train} \end{array} \right\} = P_1 + P_2 = 1026.6 + 2000 = 3026.6 \text{ lbs.}$$

Motion on a Double Inclined Plane.—Let $A B C$, $D B C$, be the two planes placed back to back, and let W_1 , be the ascending, and W_2 , the descending load, these loads being connected by a weightless rope passing over a frictionless and weightless pulley at B . We require to determine the motion—i.e., the *acceleration* of the bodies, and the tension in the connecting rope.

Let α_1, α_2 = Inclinations of planes $A B C$, $D B C$ respectively.

„ μ_1, μ_2 = Coefficients of friction between W_1, W_2 , and their respective planes.

„ F_1, F_2 = Frictional resistances in the two cases.

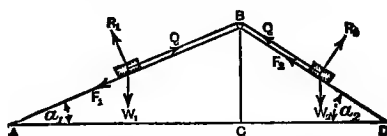
„ P_1, P_2 = Effective forces acting on W_1, W_2 respectively in causing motion.

„ Q = Tension in connecting rope.

„ a = Acceleration due to effective forces P_1, P_2 .

Then, the effective force causing the *upward* motion of W_1 , is:—

$$P_1 = Q - W_1 \sin \alpha_1 - F_1.$$



DOUBLE INCLINED PLANE.

Similarly, the effective force in causing the *downward* motion of W_2 , is:—

$$P_2 = W_2 \sin \alpha_2 - Q - F_2.$$

But, $F_1 = \mu_1 W_1 \cos \alpha_1.$

And, $F_2 = \mu_2 W_2 \cos \alpha_2.$

∴ $P_1 = Q - W_1 (\sin \alpha_1 + \mu_1 \cos \alpha_1), \dots \dots \dots (1)$

And, $P_2 = W_2 (\sin \alpha_2 - \mu_2 \cos \alpha_2) - Q. \dots \dots \dots (2)$

Again, $P_1 = \frac{W_1}{g} a \dots \dots \dots (3)$

And, $P_2 = \frac{W_2}{g} a. \dots \dots \dots (4)$

To determine the acceleration, a :—

From, (1) + (2),

$$P_1 + P_2 = W_2(\sin \alpha_2 - \mu_2 \cos \alpha_2) - W_1(\sin \alpha_1 + \mu_1 \cos \alpha_1)$$

From, (3) + (4),

$$P_1 + P_2 = \frac{W_1 + W_2}{g} \times a.$$

$$\frac{W_1 + W_2}{g} \times a = W_2(\sin \alpha_2 - \mu_2 \cos \alpha_2) - W_1(\sin \alpha_1 + \mu_1 \cos \alpha_1),$$

$$\therefore a = \frac{W_2(\sin \alpha_2 - \mu_2 \cos \alpha_2) - W_1(\sin \alpha_1 + \mu_1 \cos \alpha_1)}{W_1 + W_2} \times g. \quad (\text{II})$$

To determine the tension in the rope :—

$$\text{Equation (1)} \div (2), \quad \frac{P_1}{P_2} = \frac{Q - W_1(\sin \alpha_1 + \mu_1 \cos \alpha_1)}{W_2(\sin \alpha_2 - \mu_2 \cos \alpha_2) - Q},$$

$$,, \quad (3) \div (4), \quad \frac{P_1}{P_2} = \frac{W_1}{W_2}.$$

$$\therefore \frac{Q - W_1(\sin \alpha_1 + \mu_1 \cos \alpha_1)}{W_2(\sin \alpha_2 - \mu_2 \cos \alpha_2) - Q} = \frac{W_1}{W_2},$$

$$\therefore (W_1 + W_2)Q = W_1 W_2(\sin \alpha_1 + \mu_1 \cos \alpha_1 + \sin \alpha_2 - \mu_2 \cos \alpha_2)$$

$$\therefore Q = \frac{W_1 W_2(\sin \alpha_1 + \sin \alpha_2 + \mu_1 \cos \alpha_1 - \mu_2 \cos \alpha_2)}{W_1 + W_2} \quad (\text{III})$$

We shall show how these formulæ are modified to suit some particular cases, but the student should try to prove each particular case independently of the general case just demonstrated.

CASE I.—Suppose the planes to be equally inclined to the horizon, and equally rough, so that $\alpha_1 = \alpha_2 = \alpha$, and $\mu_1 = \mu_2 = \mu$; then, from equation (II) :—

$$a = \frac{W_2(\sin \alpha - \mu \cos \alpha) - W_1(\sin \alpha + \mu \cos \alpha)}{W_1 + W_2} g,$$

$$\therefore a = \frac{(W_2 - W_1) \sin \alpha - \mu (W_2 + W_1) \cos \alpha}{W_1 + W_2} g \quad \dots \quad (\text{II}_a)$$

From equation (III),

$$Q = \frac{2 W_1 W_2 \sin \alpha}{W_1 + W_2} \dots \dots \dots (\text{III}_a)$$

CASE II.—Let the planes be equally inclined, and smooth; so that $\alpha_1 = \alpha_2 = \alpha$, and $\mu_1 = \mu_2 = 0$; then, from equation (II):—

$$a = \frac{W_2 \sin \alpha - W_1 \sin \alpha}{W_1 + W_2} g,$$

$$\therefore a = \frac{(W_2 - W_1) \sin \alpha}{W_1 + W_2} g \quad \dots \dots \dots (II_b)$$

And from equation (III),

$$Q = \frac{2 W_1 W_2 \sin \alpha}{W_1 + W_2} \quad \dots \dots \dots (III_b)$$

Equations (III_a) and (III_b) show that the degree of roughness of the planes does not affect the tension in the rope, when the planes are equally inclined to the horizon.

CASE III.—Let the plane, AB, be horizontal, and $\mu_1 = \mu_2 = \mu$, and suppose W_2 by falling vertically to drag W_1 along AB. In this case $\alpha_1 = 0$, and $\alpha_2 = 90^\circ$; then, from equation (II):—

$$a = \frac{W_2 (\sin 90^\circ - \mu \cos 90^\circ) - W_1 (\sin 0 + \mu \cos 0)}{W_1 + W_2} g,$$

$$\therefore a = \frac{W_2 - \mu W_1}{W_1 + W_2} g \quad \dots \dots \dots (II_c)$$

And from equation (III),

$$Q = \frac{W_1 W_2 (1 + \mu)}{W_1 + W_2} \quad \dots \dots \dots (III_c)$$

CASE IV.—In the previous case, let the horizontal plane be smooth, so that, $\mu = 0$:—

$$\text{Then,} \quad a = \frac{W_2}{W_1 + W_2} g \quad \dots \dots \dots (II_d)$$

$$\text{And,} \quad Q = \frac{W_1 W_2}{W_1 + W_2} \quad \dots \dots \dots (III_d)$$

CASE V.—Suppose the weights to be suspended over the frictionless and weightless pulley B, and the parts of the rope to hang vertically.

In this case, $\alpha_1 = \alpha_2 = 90^\circ$; $\mu_1 = \mu_2 = 0$. then:—

From equation (II),

$$a = \frac{W_2 - W_1}{W_1 + W_2} g \quad \dots \quad (II_e)$$

From equation (III),

$$Q = \frac{2 W_1 W_2}{W_1 + W_2} \quad \dots \quad (III_e)$$

These last equations are of great importance to the student of *Theoretical Mechanics*, because they enable him, by means of an Atwood's machine, to determine the value of g , at the place where the experiment is conducted.

EXAMPLE IV.—A cage weighing 1 ton is being raised from a mine with an acceleration of 10 feet per second. Find the tension in the rope. If a miner, whose weight is 150 lbs., is raised with the cage, find the pressure between him and the cage. Again, if the cage be lowered with the same acceleration, what would then be the tension in the rope, and the pressure between the man and cage?

ANSWER.—(1) *To find tension in rope during ascent of cage.*

Let W = Weight of cage = 1 ton = 2,240 lbs.

„ w = Weight of man = 150 lbs.

„ Q = Tension of rope in lbs.

„ a = Acceleration of cage = 10 ft. per sec. per sec.

Then, neglecting the weight of the rope, and in the meantime that of the man, we get:—

$$\text{Effective pull causing motion} = P = Q - W.$$

But, by the *Second Law of Motion*, $P = \frac{W}{g} a$.

$$\therefore Q - W = \frac{W}{g} a.$$

$$\text{Hence,} \quad Q = W \left(1 + \frac{a}{g} \right)$$

$$\therefore Q = 2240 \times \left(1 + \frac{10}{32} \right) = 2,940 \text{ lbs.}$$

That is, the tension in the rope is *greater* than the weight raised by 700 lbs.

If the weight of the miner be taken into account, we must increase W , by 150, and then we get:—

$$Q = 3136.9 \text{ lbs.}$$

(2) *To find the pressure between man and cage.*

$$\left. \begin{array}{l} \text{Pressure between man} \\ \text{and cage} \end{array} \right\} = \left\{ \begin{array}{l} \text{Weight of man + Force required} \\ \text{to accelerate his upward motion} \end{array} \right.$$

$$\begin{array}{ccc} \text{,,} & \text{,,} & = w + \frac{w}{g}a, \end{array}$$

$$\therefore w_1 = 150 + \frac{150}{32} \times 10 = 196.9 \text{ lbs.}$$

Under these circumstances he will feel heavier by 46.9 lbs.

(3) *To find tension in rope during descent of cage.*

In this case, we get:—

$$\left. \begin{array}{l} \text{Effective pull causing} \\ \text{motion} \end{array} \right\} = P = W - Q,$$

$$\text{And,} \quad P = \frac{W}{g} \times a,$$

$$\therefore W - Q = \frac{W}{g} \times a,$$

$$\therefore Q = W \left(1 - \frac{a}{g} \right)$$

$$\therefore Q = 2,240 \times \left(1 - \frac{10}{32} \right) = 1,540 \text{ lbs.}$$

That is, the tension in the rope is *less* than the weight of the cage by 700 lbs.

Similarly, it can be shown that the pressure between the man and the floor of the cage during descent, is 103.1 lbs.; or, 46.9 lbs. less than his real weight.

EXAMPLE V.—In a double inclined plane, having a rise of 1 in 20, the loaded and empty trucks run on parallel lines of rails, the connection being made by means of two ropes passing round drums at the summit of the plane. Five loaded trucks

when descending pull up an equal number of empty ones. Each empty truck weighs 5 cwts., and when loaded carries 20 cwts. of material. The diameter of the drums at the top of the incline is 8 feet, and on the same shaft is fitted a brake pulley 6 feet in diameter. The length of the inclined plane is 1 mile. Taking the coefficient of friction between the trucks and the rails at 20 lbs. per ton, but neglecting other frictional resistances; determine (1) the acceleration of the trucks and their speed at the end of one minute after starting; (2) the tension in the ropes during the free motion of the whole; and (3) the constant frictional resistance which must be exerted at the rim of the brake pulley, during the last three-eighths of the run, in order to just bring the whole to rest at the end of the journey.

ANSWER.—Using the same letters as in the text.

Let W_1 = Total weight of five empty trucks = 25 cwts.

„ W_2 = „ „ loaded „ = 125 „

„ P_1 = Effective force causing motion of W_1 .

„ P_2 = „ „ „ W_2 .

„ Q = Tension in ropes.

α = Inclination of the plane.

„ μ = Coefficient of friction = 20 lbs. per ton = $\frac{1}{112}$.

Then, $\sin \alpha = \frac{1}{20}$; and since α is small we may assume $\cos \alpha = 1$.

(1) *To find the acceleration of the trucks.*

The effective pull causing the motion of the empty trucks, is:—

$$P_1 = Q - W_1 (\sin \alpha + \mu \cos \alpha),$$

$$\therefore P_1 = Q - 25 \left(\frac{1}{20} + \frac{1}{112} \times 1 \right) = Q - \frac{165}{112} \text{ cwt.} \quad (1)$$

The effective pull causing the motion of the loaded trucks, is:—

$$P_2 = W_2 (\sin \alpha - \mu \cos \alpha) - Q,$$

$$„ = 125 \left(\frac{1}{20} - \frac{1}{112} \times 1 \right) - Q = \frac{575}{112} - Q \text{ cwt.} \quad (2)$$

Again, by *Second Law of Motion*, we get:—

$$P_1 = \frac{W_1}{g} \times a = \frac{25}{32} \times a \text{ cwt.} \quad (3)$$

$$P_2 = \frac{W_2}{g} \times a = \frac{125}{32} \times a \text{ cwt.} \quad (4)$$

$$(1) + (2), \quad P_1 + P_2 = \frac{575}{112} - \frac{165}{112} = \frac{205}{56} \text{ cwt.}$$

$$(3) + (4), \quad P_1 + P_2 = \left(\frac{25}{32} + \frac{125}{32} \right) \times a = \frac{150}{32} a \text{ cwt.}$$

$$\therefore \quad \frac{150}{32} \times a = \frac{205}{56}$$

$$\therefore \quad a = 0.78 \text{ ft. per sec. per sec.}$$

That is, the trucks move with an acceleration of 0.78 foot per second per second.

At the end of one minute from starting the speed would be:—

$$v = a t = .78 \times 60 = 46.8 \text{ ft. per sec.}$$

Or, at the end of one minute they would be moving with a speed somewhat greater than 30 miles per hour.

(2) *To find tension in the ropes.*

Since we have assumed that the machinery at the top of the incline offers no resistance to the motion, it is evident that the tension in each rope will be the same. Hence:—

$$(1) \div (2), \quad \frac{P_1}{P_2} = \frac{Q - \frac{165}{112}}{\frac{575}{112} - Q} = \frac{112 Q - 165}{575 - 112 Q}$$

$$(3) \div (4), \quad \frac{P_1}{P_2} = \frac{W_1}{W_2} = \frac{25}{125} = \frac{1}{5}$$

$$\therefore \quad \frac{1}{5} = \frac{112 Q - 165}{575 - 112 Q}$$

$$\therefore \quad Q = \frac{1400}{672} = 2.08 \text{ cwt.}$$

(3) *To find the frictional resistance at the rim of the brake pulley in order to bring the trucks to rest at the end of the run.*

Here we have first to obtain the speed of the trucks at the instant when the brake is applied, and then find the retardation or negative acceleration necessary to bring the trucks to rest at the desired place.

The velocity v , of the trucks at the instant when the brake is applied is given by the formula :—

$$v^2 - v_1^2 = 2 a s.$$

Where v_1 = Initial velocity = 0 in this case.

„ a = Acceleration just found = 0.78 ft. per sec. per sec.

„ s = Distance traversed = $\frac{5}{8}$ mile.

The acceleration during the application of the brake may be found by the same formula. In this case, however, the initial velocity is v , and the final velocity is zero.

Let a_1 = Acceleration of the trucks during the application of the brake.

„ s_1 = Distance traversed = $\frac{3}{8}$ mile.

Then, before the brakes are applied :—

$$v^2 - 0^2 = 2 a s$$

$$\text{Or,} \quad v^2 = 2 a s.$$

And after the brakes have been applied :—

$$0^2 - v^2 = 2 a_1 s_1,$$

$$\text{Or,} \quad v^2 = -2 a_1 s_1.$$

$$\therefore a_1 = -\frac{a s}{s_1} = -\frac{.78 \times \frac{5}{8}}{\frac{3}{8}} = -1.3 \text{ ft. per sec. per sec.}$$

(4) *To determine the tensions in the two ropes.*

These will not now be equal as when the motion was free. The tension in the rope coming on to the drum will be much less than before, whilst that on the other rope will be greater.

Let Q_1, Q_2 = Tensions in the ropes attached to the empty and loaded trucks respectively.

Then the effective pull P_1 , causing the motion of the ascending trucks is as before :—

$$P_1 = Q_1 - W_1 (\sin \alpha + \mu \cos \alpha)$$

But,
$$P_1 = \frac{W_1 a_1}{g}$$

$$\begin{aligned} \therefore Q_1 &= W_1 \left\{ \sin \alpha + \mu \cos \alpha + \frac{a_1}{g} \right\} \\ „ &= 25 \left\{ \frac{1}{20} + \frac{1}{112} \times 1 + \frac{1.3}{32} \right\} \text{ cwt.}, \\ „ &= \frac{25 \times 20.5}{1120} \text{ cwt.} = 51.25 \text{ lbs.} \end{aligned}$$

Similarly, the tension Q_2 , in the rope attached to the loaded trucks is,

$$\begin{aligned} Q_2 &= W_2 \left\{ \sin \alpha - \mu \cos \alpha - \frac{a_1}{g} \right\} \\ „ &= 125 \left\{ \frac{1}{20} - \frac{1}{112} \times 1 + \frac{1.3}{32} \right\} \text{ cwt.} \\ „ &= \frac{125 \times 91.5}{1120} \text{ cwt.} = 1143.75 \text{ lbs.} \end{aligned}$$

The difference in the tensions in the two ropes is caused by the resistance offered by the brake. Hence, the resultant couple due to this difference in the tension must be balanced by the couple at the brake wheel.

Let F = Frictional resistance at the rim of the brake wheel.

„ R = Radius of the drums = 4 ft.

„ r = Radius of brake wheel = 3 ft.

Then, $F \times r = (Q_2 - Q_1) \times R$,

$$\therefore F \times 3 = (1143.75 - 51.25) \times 4,$$

$$\therefore F = 1456.7 \text{ lbs.}$$

Energy.—If we raise a body of W lbs. weight through a vertical height of h feet from some given datum level, we confer upon that body the capability of doing work equal to Wh ft.-lbs. For, in raising the body we expend Wh ft.-lbs. of work, and if it be allowed to return to its original level it will give out an equal amount of work.

Again, we have seen that if a body be in motion and its speed reduced, some force must have acted upon it in bringing about this change of state. Further, this resisting force must have been overcome through some distance, and, therefore, work is expended. Thus, a body in motion is capable of doing work, the measure of which is the work done against a resisting force or forces in bringing the body to rest.

This capability of doing work which a body possesses in virtue of its position or condition is called **Energy**. Hence, we have the following :—

DEFINITION.—The energy of a body is its capability of doing work in virtue of its Position, Condition, or Motion.

It is usual to distinguish between that form of energy due to the position or state of a body, and that due to its motion. To the former the term **Potential** is applied, and to the latter **Kinetic**. This distinction may be stated in the form of a definition.

DEFINITIONS.—**Potential Energy** is that form of energy which a body possesses in virtue of its Position or Condition.

Kinetic Energy is that form of energy which a body possesses in virtue of its Motion.

Thus, a raised weight, such as the weight of a clock, or the monkey of a pile-driving engine, has *potential* energy in virtue of its *position*. In the first case the slowly falling weight gives up its stored energy to the mechanism of the clock in overcoming frictional resistances, and thus keeps the clock going, the pendulum being simply a regulator or governor. In the second case, the monkey is allowed to fall freely and its energy is employed in forcing the pile into the ground at the instant of the blow. Similarly, the water in a mill dam possesses potential energy due to its *position* relatively to the water wheel. Again, a stretched helical spring, or coiled spiral spring such as is used in watches and clocks, possesses potential energy due to its *stretched condition*. A lump of coal, or gunpowder, has potential energy in virtue of its *chemical condition*; a magnet has potential energy in virtue of its *magnetic condition*; and the steam in a boiler has potential energy in virtue of its *heat condition*, and so on.

When the monkey of the pile driver is at the top of its stroke its energy is entirely in the potential form. When it is descending it is evident that its potential energy is rapidly decreasing whilst its kinetic energy is increasing.

Neglecting frictional and other resistances, the Principle of the Conservation of Energy (see Lecture IV., Vol. I.) asserts that—

The Loss of Potential Energy = The Gain of Kinetic Energy.

Consequently, at the instant when the monkey strikes the head of the pile, the energy of the monkey is wholly Kinetic. The work done in driving the pile into the ground is immediately derived from the kinetic energy of the falling weight, but the whole of this energy is not thus employed, for the faces of the pile and monkey have been heated by the blow. This shows that part of the energy stored in the falling weight has been transformed into heat energy. Further, at the instant of striking, a loud noise is heard, which shows that there is also a transformation into sound energy. Thus, energy appears under many different forms, such as mechanical, electrical, chemical, heat, light, sound, &c., and can, by suitable arrangements, be changed from one kind into any of the others. In nature all is change or transformation, but there is no annihilation; so what appears as a loss to the engineer simply means change into some other form which he does not desire, but which he has no power to entirely prevent.

Expression for Kinetic Energy.—We have already seen, that the expression for mechanical potential energy is :—

$$\text{Potential Energy} = E_p = W h$$

Where, W = Weight of body,

And, h = Height of body above zero level.

It now remains to determine the expression for kinetic energy.

First, take the case of the raised weight whose potential energy in its highest position is $W h$, and suppose it to fall freely. Its kinetic energy at the instant when it strikes the ground is :—

$$E_k = W h.$$

But, if v = velocity at that instant, we have :—

$$v^2 = 2 g h,$$

$$\text{Or,} \quad h = \frac{v^2}{2 g}.$$

$$\therefore E_k = \frac{W v^2}{2 g} \quad \dots \dots \dots \text{(IV)}$$

Thus, if the monkey of the pile driver weigh 10 cwts., and reaches the pile with a velocity of 40 feet per second, it has kinetic energy :—

$$E_k = \frac{10 \times 112 \times 40^2}{2 \times 32} = 28,000 \text{ ft.-lbs.}$$

If the pile be driven 3 inches into the ground at each blow, what is the *mean* resistance offered to its motion, supposing there are no losses from heating, &c.?

Let R_m = Mean resistance of ground in lbs.

„ s = Distance in feet through which the pile is driven.

Then, $\left. \begin{array}{l} \text{Work done in driv-} \\ \text{ing the pile} \end{array} \right\} = \left\{ \begin{array}{l} \text{Work given out by monkey.} \end{array} \right.$

But, $\left. \begin{array}{l} \text{Work given out by} \\ \text{monkey} \end{array} \right\} = \left\{ \begin{array}{l} \text{Kinetic energy at instant of} \\ \text{striking the pile} + \text{Work} \\ \text{done in falling through 3"}. \end{array} \right.$

$$\therefore R_m \times s = \frac{W v^2}{2g} + W \times s,$$

$$\text{i.e., } R_m \times \frac{3}{12} = 28,000 + 10 \times 112 \times \frac{3}{12},$$

$$\therefore R_m = 113,120 \text{ lbs.} = 50.5 \text{ tons.}$$

Energy Equations.—The expression for the kinetic energy given in equation (IV) is perfectly general, and therefore independent of the manner in which the velocity, v , is acquired. That is to say, if a body of weight, W , be moving with a velocity, v , in any direction whatever, its kinetic energy is still given by the equation :—

$$E_k = \frac{W v^2}{2g}.$$

For, manifestly, the direction of motion cannot in any way affect its energy state, other things being equal. Nevertheless, we can deduce the expression from more general considerations as follows :—

Let a body of weight, W , have its velocity changed in magnitude from v_1 to v_2 by a constant force P , acting through a distance s . Then—

$$\left. \begin{array}{l} \text{Change of body's kinetic} \\ \text{energy} \end{array} \right\} = \left\{ \begin{array}{l} \text{Work done by or against} \\ \text{the force} \end{array} \right.$$

And, *Work done by or on P* = $P \times s$.

But, by equation (I), $P = \frac{W a}{g}$,

And, by equation (VI), Lecture X.,

$$s = \frac{v_2^2 - v_1^2}{2a},$$

$$\therefore P \times s = \frac{W(v_2^2 - v_1^2)}{2g} = \text{change of } E_K. (V)$$

If the body start from rest, and have a final velocity, v , we get:—

$$\text{Change of } E_K = \frac{W v^2}{2g} = \text{Total } E_K \text{ in body,}$$

which is just the same result as that given by equation (IV).

Next, take the case of a body moving with uniform velocity, v , against some constant resistance, F , which resistance may be frictional or otherwise. To maintain this constant speed a force equal to F must act on the body in opposition to the resistance; but no part of this force is employed in changing the kinetic energy of the body, since, by supposition, no change occurs in its velocity. The kinetic energy of the body is constantly $= \frac{W v^2}{2g}$, and the *Work done against resistances* = $F s$.

If, now, some other force, P , acts on the body in the direction of motion, the velocity will change, and, therefore, the energy of the body will also change.

Let Q = Resultant force acting on body = $P \sim F$.

„ v_1, v_2 = Velocities of body before and after action of P .

„ s = Distance through which body moves under P .

$$\text{Then, } Q \times s = F \times s + \frac{W(v_2^2 - v_1^2)}{2g} \quad . . . (VI)$$

This is a very general equation of energy, and is sometimes stated thus:—

Energy exerted = Work done + Change in Kinetic Energy.

EXAMPLE VI.—The height and length of an inclined plane are 20 feet and 100 feet respectively: a body weighing 100 lbs. is placed at the top of the plane and allowed to slide along its whole length; the coefficient of friction between the plane and

the body is 0.15; how many units of work (foot-pounds) are accumulated in the body, and what is its velocity when it reaches the foot of the plane? (You may assume the pressure on the plane equal to the weight of the body). (S. & A. Adv. Exam.)

ANSWER.—Let F = Resultant force urging body down the plane.

Then, $F = W \sin \alpha - \mu R = W \sin \alpha - \mu W$, very approximately,

$$,, = 100 \left(\frac{20}{100} - 0.15 \right) = 5 \text{ lbs.}$$

When body reaches the foot of the plane, we have:—

$$E_K = F \times l = 5 \times 100 = 500 \text{ ft.-lbs.}$$

Let v = Velocity at foot of plane.

$$\text{Then, } \frac{W v^2}{2g} = E_K$$

$$\text{i.e., } \frac{100 \times v^2}{2 \times 32} = 500,$$

$$\therefore v = 17.9 \text{ ft. per sec.}$$

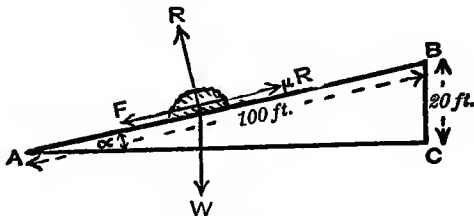


DIAGRAM ILLUSTRATING EXAMPLE VI.

The kinetic energy at the foot of the plane could be obtained immediately from equation (VI), thus:—

$$\text{Energy exerted} = W \sin \alpha \times l = 100 \times \frac{20}{100} \times 100 = 2000 \text{ ft.-lbs.}$$

$$\text{Work done} = \mu W l = 0.15 \times 100 \times 100 = 1500 \text{ ft.-lbs.}$$

$$\text{Change in } E_K = \text{Energy at foot of plane} = \frac{W v^2}{2g}.$$

∴ Energy exerted = Work done + Change in E_k .

$$\therefore 2000 = 1500 + \frac{W v^2}{2g}.$$

$$\therefore \left. \begin{array}{l} E_k \text{ at foot of} \\ \text{plane} \end{array} \right\} = \frac{W v^2}{2g} = 500 \text{ ft.-lbs.}$$

$$\therefore \text{also, } v = \sqrt{\frac{500 \times 2 \times 32}{100}} = 17.9 \text{ ft. per sec.}$$

EXAMPLE VII.—Show by a diagram the amount of work done in *slowly compressing* a spiral spring through 6 inches, supposing the spring to shorten 1 inch for every 100 lbs. pressure. If a weight of 100 lbs. *falls* from a height of 4 feet on the top of the spring, how much will it be compressed? (S. & A. Adv. Exam., 1892.)

ANSWER.—As explained in Lecture II., Vol. I., the diagram of work done in slowly compressing a spiral spring, is a right angled triangle whose base represents the compression produced, and its perpendicular side the force required to produce that compression.

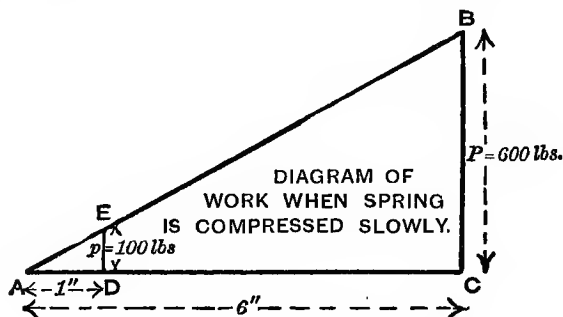


DIAGRAM ILLUSTRATING EXAMPLE VII.

Let A B C represent the diagram of work done in slowly compressing the spring.

Where, A C = Compression produced = 6".

And, C B = Force required = P.

Let, D E = Force required to compress spring 1",
i.e., D E = p = 100 lbs.

Then, $P : p = CB : DE = AC : AD$.

That is, $P : 100 = 6'' : 1''$.

$\therefore P = 600 \text{ lbs.}$

$\therefore \text{Work done} = \frac{1}{2} P \times L = \frac{1}{2} \times 600 \times 6'' = 1800 \text{ in.-lbs.}$

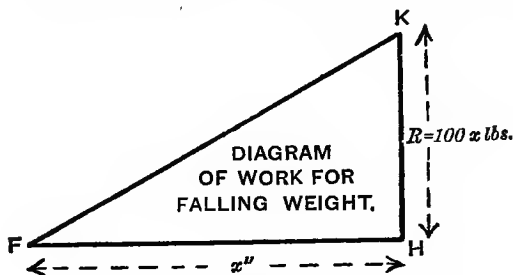


DIAGRAM ILLUSTRATING EXAMPLE VII.

Next, suppose the spring to be compressed by a weight which falls from a height of 4 feet.

Let x = Number of *inches* by which spring is compressed by falling weight.

Then, $48 + x$ = Number of inches through which weight actually falls.

Since a force of 100 lbs. is required to compress the spring 1 inch, a force $R = 100 x$ lbs. will be required to compress it x inches.

But,

$$\left. \begin{array}{l} \text{Work done in compress-} \\ \text{ing spring} \end{array} \right\} = \left\{ \begin{array}{l} \text{Work done by falling} \\ \text{weight.} \end{array} \right.$$

$$\therefore \frac{1}{2} R \times x = W (48 + x).$$

$$\therefore \frac{1}{2} \times 100 x \times x = 100 (48 + x).$$

$$\text{That is,} \quad x^2 = 2x + 96.$$

$$\text{Or,} \quad x = 10.85 \text{ inches.}$$

EXAMPLE VIII.—A blowing-fan 30 inches in diameter revolves at a speed of 1,000 revolutions per minute, and propels the wind with a velocity equal to $\frac{7}{8}$ of the velocity of the tips of the vanes; the wind is driven through a pipe having a sectional area

of 200 square inches. Neglecting the power that is required to overcome friction, will you show the amount of power which is required to force the above quantity of air? Work it out in arithmetic, either by the law of falling bodies or in any better way that may suggest itself to you. (S. & A. Hons. Exam.)

ANSWER.—This is simply a question on work and energy.

Let W = Weight of air expelled from fan *per second*.

„ A = Sectional area of delivery pipe of fan = 200 sq. ins.

„ v = Velocity of air as it leaves tips of vanes.

Then, $v = \frac{7}{8} \times \text{Velocity of tips of vanes}$

$$„ = \frac{7}{8} \times \pi d n$$

$$„ = \frac{7}{8} \times \frac{22}{7} \times \frac{30}{12} \times \frac{1000}{60} = 114.6 \text{ ft. per sec.}$$

$$\left. \begin{array}{l} \text{Volume of air expelled} \\ \text{per second} \end{array} \right\} = A v$$

$$„ \quad „ \quad = \frac{200}{144} \times 114.6 = 159.16 \text{ cub. ft. per sec.}$$

By calculation from the density of air, it can be shown that 13 cubic feet of air at atmospheric pressure weigh 1 lb. very nearly, and assuming this, we get:—

$$\text{Weight of } 159.16 \text{ cub. ft.} = W = \frac{159.16}{13} = 12.24 \text{ lbs. nearly.}$$

$$\text{Hence, Work done per sec.} = \text{Energy exerted} = \frac{W v^2}{2 g}$$

$$„ \quad „ \quad = \frac{12.24 \times 114.6^2}{2 \times 32} \text{ ft.-lbs. per sec.}$$

$$„ \quad „ \quad = \frac{12.24 \times 114.6^2 \times 60}{2 \times 32} \text{ ft.-lbs. per min.}$$

$$\therefore \text{H P. exerted} = \frac{12.24 \times 114.6^2 \times 60}{2 \times 32 \times 33,000} = 4.57 \text{ nearly.}$$

LECTURE XI.—QUESTIONS.

1. Define momentum, and state how it is measured. State and explain, by aid of examples, Newton's three laws of motion. A shot weighing half a ton is fired from a 100-ton gun with a velocity of 2,000 feet per second. Neglecting the weight of the powder, find the velocity of the gun's recoil. *Ans.* 10 feet per second.

2. A man weighing 140 lbs. descends in a lift with an acceleration equal to $\frac{1}{4}g$. What pressure does he exert on the floor of the lift? How would your answer differ if the lift were ascending instead of descending? *Ans.* 122.5 lbs.; 157.5 lbs.

3. A railway train, exclusive of engine, weighs 150 tons, and in starting along a level line from rest it attains a speed of 30 miles an hour in 5 minutes. What has been the mean pull between the engine and train, the resistance being taken at 10 lbs. per ton? *Ans.* 3,040 lbs.

4. A locomotive and its train weigh 220 tons, and the frictional resistance at all speeds may be taken at 2,000 lbs. If the tractive force of the engine is constantly 3,500 lbs., find in what time from starting the train can attain a speed of 40 miles per hour (1) on a level line, and (2) going down an incline of 1 in 220. Find, also, the distance travelled in both cases in attaining the above speed. *Ans.* (1) 10 minutes 2 seconds, 3.34 miles; (2) 4 minutes 1 second, 1.34 miles.

5. In a steam engine, the piston, which is 40 inches diameter and weighs 2,000 lbs., comes off the rod just as it is commencing its inward stroke. The mean steam pressure is 50 lbs. per square inch. Find the velocity with which the piston will strike the cover at the opposite end of the cylinder, the stroke being 4 feet. *Ans.* 89.7 feet per second.

6. One end of a string is fixed; it then passes over a movable pulley to which a weight, W , is attached. The string then passes over a fixed pulley, and a smaller weight, w , is attached to its other end, all three sections of the string being vertical. Show that, neglecting the weights of the pulleys, the acceleration with which W descends is $\left(\frac{W-2w}{W+4w}\right)g$.

Verify this result (1) when w is small compared with W , and (2) when W is small compared with w . (Wool. Roy. Milly. Acad. Exams.)

7. It is very evident that a railway train requires a considerable amount of force to set it in motion, but there is a popular notion existing that a less amount of power or force is required to bring the same train to a state of rest. Will you explain clearly the natural principles upon which the whole case depends, and compare the force necessary both for giving motion to the train and in producing the opposite condition? (S. & A. Hons. Exam.)

8. Distinguish between work and energy, and between potential and kinetic energy. Give examples of both forms of energy. State the principle of the conservation of energy, and show its connection with the axiom that "perpetual motion" is impossible. A simple pendulum is pulled aside till its heavy bob is raised h inches vertically, and then let go. Find its velocity when it passes its lowest point. *Ans.* $\sqrt{2gh}$.

9. Prove the formula which gives the number of units of work stored up in a given weight when moving with a given velocity. A weight of

100 lbs. is moving with a velocity of 64 feet per second, how many foot-pounds of work have been expended in producing this result? (S. & A. Adv. Exam.) *Ans.* 6,400 foot-pounds.

10. A hammer head weighs 5 tons and reaches the anvil with a velocity of 10 feet per second; what amount of energy measured in foot-pounds, is stored up in the hammer at the instant of the blow? *Ans.* 17,500 foot-pounds.

11. The head of a steam hammer weighs 10 cwts., and has a fall of 8 feet. If it indent the iron on which it falls by 1 inch, find the mean force exerted on the iron during compression. *Ans.* 970 cwts.

12. Of two steam hammers, one weighs 5 tons and reaches the anvil with a velocity of 10 feet per second, and the other weighs 10 tons and reaches the anvil at a velocity of 5 feet per second; will you compare and distinctly characterise the conditions of the blow of each of the two hammers? (S. & A. Hons. Exam.) *Ans.* 2:1.

13. Referring to a steam hammer, in which steam is admitted above the piston to assist gravitation, will you describe the combination of forces at work in producing the blow, and, as far as you may be able, the nature of the blow as depending on velocity and mass or weight of the hammer at the moment of impact? (S. & A. Hons. Exam.)

14. What do you understand by energy, and how is it measured? The head of a steam hammer weighs 50 cwts., steam is admitted on the under side for lifting only, and there is a drop of 5 feet. What will be the average compressive force exerted during a blow from this hammer, on the supposition that the duration of the blow—that is, the time during which the hammer is compressing the iron under operation—is $\frac{1}{8}$ second? *Ans.* 2,285 cwts.

15. The monkey of a pile driver weighs 15 cwts., and the drop is 6 feet. The blow causes the pile to go down through 4 inches; what is the frictional resistance of the earth? *Ans.* 285 cwts.

16. Compare energies expended in pile driving by a ram or monkey of 1 ton falling 20 feet, with that of a weight of 2 tons falling 10 feet. If one blow of the former moves the pile 9 inches, what is the average resistance that is opposed to its motion? *Ans.* (1) E_k is the same in each case; (2) 27·7 tons.

17. Two bodies, weighing 5 lbs. and 3 lbs. respectively, are connected by a perfectly flexible weightless string which passes over a smooth pulley. The heavier body draws up the lighter. When it has fallen through 5 feet, what is the kinetic energy of the bodies and the velocity? ($g = 32$.) Determine also the acceleration of the system, and the tension in the string. *Ans.* (1) 10 foot-pounds; (2) 8·94 feet per second; (3) 8 feet per second per second; (4) 3·75 lbs.

18. State Newton's third law of motion, and give his illustrations of it. Weights of 5 and 11 lbs. are connected by a weightless thread. The latter is placed on a smooth horizontal table, while the former hangs over the edge. If the bodies are then allowed to move under the action of gravity, what is the tension of the thread? Find, also, the acceleration produced, and the kinetic energy of the system at the end of 4 seconds. *Ans.* (1) 3·44 lbs., (2) 10 feet per second per second, (3) 400 foot-pounds.

19. A train of locomotive and carriages weighs 60 tons. If it be supposed to run down an incline of 1 in 265 for $7\frac{1}{2}$ miles, starting with zero velocity, unopposed by anything but its own inertia, and unaccelerated by anything but its own weight; what would be its velocity, its momentum, and its kinetic energy at the foot of the incline? *Ans.* 97·8 feet per second; 5,870 units of momentum; 8,966 foot-tons.

20. What meaning do you attach to the phrase *horse-power*? A fire-engine pump is provided with a nozzle, the sectional area of which is 1 square inch, and the water is projected through the nozzle with a velocity of 130 feet per second; find the horse-power of the engine required to drive the pump, irrespective of the loss by resistance of the working parts. The weight of a cubic foot of water is $62\frac{1}{2}$ lbs. (S. & A. Hons. Exam.) *Ans.* 27.1 H.P.

21. State Newton's second law of motion. Explain briefly how the measure of force is derived from this law. In the equation $P = mf$, in what units is P , when the units of mass, distance, and time are a pound, a foot, and a second? (S. & A. Adv. Theor. Mechs. Exam., 1896.)

22. A steam engine is employed to raise coals, and it is calculated that in order to set in motion the winding drums, flywheel, cages, ropes, &c., which are concerned in the motion, it has to do the work of imparting a linear velocity of 36 feet per second, to 60 tons of material in *half-a-minute* at each lift. What effective horse-power is consumed in overcoming the inertia of the aggregate weight of 60 tons, and in setting up the velocity, estimated as above stated, in the time assigned? (S. & A. Hons. Exam.) *Ans.* (i.) 146.6 H.P.; (ii.) 5.5 H.P. nearly.

23. In lifting water into a tender by a scoop running along a trough while the train is going at rapid speed, the height of the lift is $7\frac{1}{2}$ feet. What speed in miles per hour will just cause the water to be lifted through that height? (S. & A. Hons. Exam.) *Ans.* 15 miles per hour nearly.

24. A vertical pipe, carried by the tender of a locomotive engine, and terminating in a scoop with a flat mouth, picks up water from a trough laid on a railway. If the speed of the engine and tender be 22 miles per hour, find the height to which the water will rise in the pipe. Upon what theory do you proceed? *Ans.* 16.3 feet.

25. A hammer head of $2\frac{1}{2}$ lbs., moving with a velocity of 50 feet per second, is stopped in 0.001 second. What is the average force of the blow? What do you mean by this average? What is the difference between a time average and a space average? When are they the same? *Ans.*

26. A body of 4 lbs. moving with a speed of 20 feet per second overtakes one of 200 lbs. moving at 2 feet per second in the same direction. When the collision events are finished (friction stilling the relative motions) and both bodies go on together, what is their common velocity? What mechanical energy has been lost?

27. A body of 200 lbs. is acted on by a force F , which alters. No other force acts on the body. When the body has passed through the distance x feet, the force in pounds is as follows:—

x	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
F	20	21	21	20	19	18.5	18.0	13.5	9	4.5	0

Using either a graphical or arithmetical method, find—

- The average force acting on the body through this total distance of 1 foot.
- The work done upon the body from $x = 0$ to $x = .4$.
- The answer to (b) being the kinetic energy added to the body; if the velocity was 0 when x is 0, what is the velocity when $x = .4$? *Ans.*

28. Prove the formula for the kinetic energy of a moving body. A ball weighing 1 lb. moving at 1,200 feet per second passes through a plate of iron in 0.002 second, and its velocity is reduced to 200 feet per second. Find the work done in passing through the plate, and the average force during the time of its passage.

29. A machine is found to have 300,000 foot-pounds stored in it as kinetic energy when its main shaft makes 100 revolutions per minute. If the speed changes to 98 revolutions per minute, how much kinetic energy has it lost? A similar machine (that is, made to the same drawings but on a different scale) is made of the same material but with all its dimensions 20 per cent. greater. What will be its store of energy at 70 revolutions per minute? What energy will it store in changing from 70 to 71 revolutions per minute? (B. of E. Adv. & H., Part I., 1901.)

30. A body weighing 1,610 lbs. is lifted vertically by a rope, there being a damped spring balance to indicate the pulling force, F lbs., of the rope. When the body has been lifted x feet from its position of rest, the pulling force was automatically recorded as follows:—

x	0	11	20	34	45	55	66	76
F	4,010	3,915	3,763	3,532	3,366	3,208	3,100	3,007

(using squared paper). Find approximately the work done on the body when it has risen 70 feet. How much of this is stored as potential energy, and how much as kinetic energy? What is then the velocity of the body? Find the velocity, v , feet per second for values of x of 10, 30, 50, 70, and draw a curve showing the probable values of v for all values of x up to 80. In what time does the body get from $x = 45$ to $x = 55$?

(B. of E. Adv. & H., Part I., 1901.)

31. An electric tramcar, loaded with 52 passengers, weighs altogether 10 tons. On a level road it is travelling at a certain speed. For the purpose of finding the tractive force, the electricity is suddenly turned off, and an instrument shows that there is a retardation in speed: how much will this be if the tractive force was 315 lbs.? If the tractive force is found on several trials to be on the average—

342 lbs.	when the speed is 12 miles per hour,
315 "	" " " 10 "
294 "	" " " 8 "

what is the probable tractive force at 9 miles per hour?

(B. of E. Adv. & H., Part I., 1901.)

32. A car weighs 10 tons, what is its mass in engineers' units? It is drawn by the pull, P lbs., varying in the following way, t being seconds from the time of starting:—

P	1,020	980	882	720	702	650	713	722	805
t	0	2	5	8	10	13	16	19	22

The retarding force of friction is constant, and equal to 410 lbs. Plot

$P-410$ and the time, t , and find the *time average* of this excess force. What does this represent when it is multiplied by 22 seconds? What is the speed of the car at the time 22 seconds from rest? Tabulate values of speed and time, and draw a curve showing speed and time.
(B. of E., Adv & H., Part I., 1902.)

N.B.—See *Appendices B and C* for other questions and answers.

LECTURE XI.—A.M.INST.C.E. EXAM. QUESTIONS.

1. What is meant by the conservation of momentum? A fireman holds a round nozzle from which a jet of water $\frac{1}{2}$ inch in diameter is projected with a velocity sufficient to carry it to a height of 100 feet. Find the force in lbs. which he has to exert in holding the nozzle. (I.C.E., Oct., 1897.)

2. If a man coasting on a bicycle down a uniform slope of one in fifty attains a limiting speed of 8 miles per hour, what horse-power must he exert to drive his machine up the same hill at the same speed, there being no wind in either case? The weight of man and bicycle together is 200 lbs. (I.C.E., Oct., 1897.)

3. Apply the principle of the conservation of energy to find the velocity of a thin hollow circular cylinder after rolling a distance of 12 feet down a plane inclined at a slope of 1 vertical in 5 horizontal. (I.C.E., Feb., 1898.)

4. Assuming that a train may be accelerated by the application of a force equal to one-fortieth of its gross weight and be braked with a force equal to one-tenth of its gross weight, find the least time in which it may be run from one to another of two stopping stations 5,000 feet apart. What is its greatest speed during the run? (I.C.E., Feb., 1898.)

5. A hammer-head of $2\frac{1}{2}$ lbs., moving with a velocity of 75 feet per second, is stopped in 0.0007 second. What is the average force of the blow? (I.C.E., Oct., 1898.)

6. Define velocity, acceleration, momentum, mass or inertia, force, impulse, rate of change of momentum per second, kinetic energy. In every case, when you make a statement relating to linear or translational motion, make the analogous statement about angular motion.

(I.C.E., Oct., 1898.)

7. A body of 30 lbs. moves towards the south at 30 feet per second, in 2 minutes it moves towards the south-west at 40 feet per second—what is the added velocity? Find the average acceleration. What constant force would produce the change? (I.C.E., Oct., 1898.)

8 The pull, F , in lbs. on a tramcar was registered when the car was at the following distances x from a certain point. Use squared paper to find

x	0	12	20	34	50	62
F	510	480	460	450	470	490

the average pull and the work done. If the distance is passed over in 21 seconds, find the average horse-power. (I.C.E., Oct., 1898.)

9. Taking the resistance, F lb., of a bicycle on a level road to be given by $F = W \{0.002 + 0.00012 (v + w)^2\}$, where W is the weight, in lbs., of rider and machine, and v is the speed in miles per hour, w being the speed of an opposing wind; calculate what H.P. is expended in going up an incline of 1 in 80 at a speed of 10 miles an hour, the weight of the rider and machine being 180 lbs., the helping wind being at 5 miles an hour. What would this amount to going down the same slope at double the speed, the wind now opposing the rider? (I.C.E., Feb., 1899.)

10. In a steam engine the piston at the beginning of its stroke is exposed to a total pressure of 2,000 lbs., but the inertia is such that the thrust of the piston-rod at the crosshead is only 1,600 lbs. The speed of the engine

is now raised until it becomes half as great again as before, while the steam pressure is unchanged, what is the thrust of the piston-rod?

(I.C.E., *Feb.*, 1899.)

11. State Newton's second law of motion in its most useful form. Suppose that a Maxim gun delivers 250 1-oz. bullets per minute with a speed of 1,500 feet per second, what average force in lbs. weight must be provided to hold the gun still? (I.C.E., *Feb.*, 1899.)

12. A vehicle weighing 4 tons is proceeding at a rate of 10 miles an hour along a level road; the pull on it is suddenly stopped; supposing the whole resistance equivalent to 500 lbs. applied to the rim of one of the wheels 4 feet diameter, calculate how far the vehicle will run before stopping. (I.C.E., *Oct.*, 1899.)

13. An engine of 10 B.H.P. drives the axle of a vehicle at one-fifth the angular velocity of its own shaft; the driving wheels are 4 feet diameter, and the speed of the vehicle is 8 miles per hour, find the equivalent resultant pull on the driving axle. (I.C.E., *Oct.*, 1899.)

14. Give Newton's second law of motion. Calculate the force necessary to increase the velocity of a body weighing 100 lbs. by 10 feet per second in 4 seconds, supposing the acceleration uniform. (I.C.E., *Feb.*, 1900.)

15. A mass whose weight is 50 lbs., moving at the rate of 15 feet per second, is acted upon for 20 seconds by a force of 20 lbs.; find the distance moved during the time. (I.C.E., *Feb.*, 1900.)

16. A bicycle and rider, weighing together 180 lbs., are travelling at the rate of 10 miles per hour on the level. Supposing a brake is applied to the top of the front wheel 30 inches in diameter, and that this is the only resistance acting, how far will the bicycle travel before stopping if the pressure of the brake is 20 lbs., and the coefficient of friction 0.5? In what respect is such a bicycle brake more efficient than a brake on a vehicle with springs? (I.C.E., *Feb.*, 1900.)

17. Explain the chief units adopted in the measurement of acceleration, force, energy, and power. If a bicyclist always works at $\frac{1}{4}$ H.P. and goes 12 miles per hour on the level, find the resistance of the road, and show that, if the mass of the machine and the rider together be 12 stone, the speed on an incline of 1 in 50 will be reduced to about 5.8 miles per hour. (I.C.E., *Oct.*, 1900.)

18. A man pushes an 8-ton railway truck from rest with a uniform force of $\frac{1}{2}$ cwt. on a smooth horizontal line of rails. Neglecting all resistances, find what speed he will get up in 22 seconds. At what H.P. will he be working at the end of that time? (I.C.E., *Feb.*, 1901.)

19. A 1-oz. bullet fired horizontally, with velocity 1,000 feet, into a 1-lb. block of wood, resting on a smooth table, penetrates 2 inches and remains imbedded. With what velocity does the block move off (without rotation)? Why would the bullet have penetrated more if the block had been fixed?

(I.C.E., *Feb.*, 1901.)

20. A man weighing 140 lbs. stands on the floor of a lift. Find the pressure he exerts on the floor (a) when the lift ascends and descends with uniform velocity, (b) when it ascends with a velocity which decreases by the acceleration of $\frac{1}{4}g$, (c) when it descends with a velocity which increases at the rate of 8 feet per second per second. Under what conditions can the pressure be (i.) zero, (ii.) greater than the weight of the man? (I.C.E., *Oct.*, 1901.)

21. A cannon weighs 35 tons and the shot 1,200 lbs. The velocity of the shot on leaving the muzzle is 1,200 feet per second, find the velocity of the recoil of the cannon. If the velocity of recoil is to be destroyed while the gun moves through 3 feet, find the average resistance to be applied.

(I.C.E., *Oct.*, 1901.)

22. Distinguish between kinetic and potential energy and show how they are measured in the case of a body falling from rest. Show that if a body be let fall from rest its energy remains constant till it reaches the ground. (I.C.E., Oct., 1901.)

23. Define "impulse" and "momentum." A force acting on a mass of 10 lbs. increases its velocity in every second by 7 feet a second. Another force acting on a body of mass 25 lbs. increases its velocity in $2\frac{1}{2}$ minutes from 500 feet per second to 1,850 feet per second. Compare the forces.

(I.C.E., Feb., 1902.)

24. State exactly what is meant by "a pound mass," "a pound weight," "a poundal," and by the terms "work" and "energy." Show that if a body of mass m has a velocity v imparted to it by the action of a constant force F acting through a distance S , then the work done by the force is $\frac{1}{2} m v^2$. (I.C.E., Feb., 1902.)

25. Two bodies, P and Q , of unequal mass, are connected by a fine string passing over a frictionless pulley. Find expressions for the acceleration of the bodies, and for the tension in the string. (I.C.E., Feb., 1902.)

26. The resistance of a passenger train on the level road is 17.3 lbs. per ton, the speed being 48 miles per hour. If the total weight of engine and train is 190 tons, find the horse-power of the engine. If the train is brought to a standstill by the application of the brakes in $14\frac{1}{2}$ seconds, find the average resistance of the brakes. (I.C.E., Feb., 1902.)

27. A train increases its speed from 40 miles to 49 miles an hour while descending an incline in $4\frac{1}{2}$ minutes. Find its average acceleration. Find also the slope of the incline, taking the acceleration due to gravity to be 32 foot-second units. (I.C.E., Feb., 1902.)

28. At what horse-power must a bicyclist work when riding at 20 miles per hour on a track, the resistance of which is 1 per cent. of the total weight (180 lbs.)? If he goes 60 yards from rest before getting speed up, find the mean moment he must exert during that time. Diameter of wheel 28 inches, and geared so as to make 9 revolutions for 4 turns of the crank. (I.C.E., Oct., 1902.)

29. Explain the term "kinetic energy," and show that if the motion of a body be retarded by a resistance, the decrease of kinetic energy is equal to the work done against that resistance. At what distance from a given point must a carriage be detached from a train going at 20 miles an hour in order that it may come to rest at the given point, the brake being applied and the coefficient of friction being $\frac{1}{4}$? (I.C.E., Oct., 1902.)

30. Distinguish between the measurements of force and impulse. The head of a steam hammer weighs 50 cwts.; steam is admitted on the under side for lifting only, and there is a drop of 5 feet. What will be the velocity and momentum of the head the instant before the blow is given, if there is no resistance to the fall. If the time during which the compression of the iron takes place be $\frac{1}{10}$ second, find the average force of the blow. (I.C.E., Oct., 1902.)

31. Explain the principle of conservation of energy. Two cylindrical tanks, A and B , of, respectively, 4 square yards and 2 square yards horizontal cross section, stand on the same floor and are connected near the bottom by a narrow pipe. A at first contains 8 cubic yards of water, and B is empty. The water flows slowly into B ; find the amount of heat which will have been generated when the water has ceased to flow and it has all come to rest. (I.C.E., Feb., 1903.)

32. Two men put a railway wagon weighing 5 tons into motion by exerting on it a force of 80 lbs. The resistance of the wagon is 10 lbs. per ton, or altogether 50 lbs.; how far will the wagon have moved in 1 minute?

Calculate at what fraction of a horse-power the men are working at 60 seconds after starting. (I.C.E., Feb., 1903.)

33. State and explain fully Newton's Third Law of Motion. A 100-lb. shot leaves a gun horizontally with a muzzle velocity of 2,000 feet per second. The gun and attachments, which recoil, weigh 4 tons. Find what the resistance must be that the recoil may be taken up in 4 feet, and compare the energy of recoil with the energy of translation of the shot.

(I.C.E., Feb., 1903.)

N.B.—See *Appendices B and C* for other questions and answers.

LECTURE XII.

CONTENTS.—Energy of a Rotating Body—Moment of Inertia of a Body about an Axis—Definitions of Moment of Inertia and Radius of Gyration—Propositions I., II., and III.—Methods of Calculating Moments of Inertia—Examples I., II., and III.—Tables of Radii of Gyration of Solids and Sections—Equation of Energy for a Rotating Body—Examples IV., V., VI., and VII.—Determination of Energy of Flywheels—Centripetal and Centrifugal Forces—Definitions of Centripetal and Centrifugal Forces—Example VIII.—Straining Actions due to Centrifugal Forces—Example IX.—Questions.

Energy of a Rotating Body.—The deduction of the equation for energy of rotation is complicated by the fact that particles of the body at different distances from the axis of rotation possess different energies, due to their different linear velocities. To obtain the energy of the whole body, we must, therefore, take the sum of the energies of the various particles composing it. In general, this process must be performed by the aid of higher mathematics; and even then, only in those cases in which the bodies are of regular geometrical form.

Moment of Inertia of a Body about an Axis.—Before deducing the expression for the kinetic energy of a rotating body, it may be as well to explain certain terms and quantities which we shall have occasion to make use of.

DEFINITION.—If the mass of every particle of a body be multiplied by the square of its distance from a given axis, the sum of the products is called the Moment of Inertia of the body about that axis.

Let I = Moment of inertia of the body about a given axis.

„ m = Mass of any particle or element of body.

„ r = Distance of m from the given axis.

Then, $I = \Sigma m r^2$ (I)

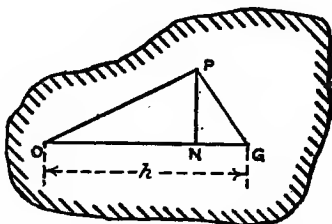
DEFINITION.—If M be the mass of a body, and k be such a quantity that $M k^2$ is its Moment of Inertia about a given axis, then k is called the Radius of Gyration of the body about that axis.

Thus, $M k^2 = I$ }
 Or, $k^2 = \frac{I}{M}$ } (II)

The following PROPOSITIONS relating to moments of inertia and radii of gyration are so important that we here give their proofs in full:—

PROPOSITION I.—If M be the mass of any body, I_G the moment of inertia about any axis through its centre of gravity, G , and I_O , that about a parallel axis through any other point, O , at a distance, h , from G , then:—

$$I_O = I_G + M h^2. \quad \dots \quad (III)$$



MOMENTS OF INERTIA ABOUT PARALLEL AXES.

Let G , and O , be the points of intersection of the axes with the plane of the paper, which is at right angles to them. Let P be any particle of mass, m . Draw PN perpendicular to OG .

Then, in triangle OPG , we get (Euc. II., 13):—

$$OP^2 = PG^2 + OG^2 - 2 OG \cdot GN.$$

Multiplying both sides by m the mass of particle at P , we get:—

$$m \cdot OP^2 = m \cdot PG^2 + m \cdot OG^2 - 2 m \cdot OG \cdot GN.$$

Repeating this process for every other particle of the body, and adding the results, we have:—

$$\Sigma m \cdot OP^2 = \Sigma m \cdot PG^2 + \Sigma m \cdot OG^2 - 2 \Sigma m \cdot OG \cdot GN.$$

But, clearly, $\Sigma m \cdot OP^2 = I_O$, and $\Sigma m \cdot PG^2 = I_G$.

And, since $OG = h = \text{constant}$,

$$\therefore \Sigma m \cdot OG^2 = OG^2 \Sigma m = h^2 M, \text{ or } M h^2.$$

Also, $2 \Sigma m \cdot OG \cdot GN = 2 OG \cdot \Sigma m \cdot GN = 2 h \Sigma m \cdot GN$.

But the quantity, $\Sigma m \cdot GN$, is the sum of the moments of the various particles about their centre of gravity, G , and is therefore zero, from the definition of the centre of gravity.

$$\therefore 2 \Sigma m \cdot O G \cdot G N = 2 h \Sigma m \cdot G N = 0.$$

$$\therefore I_0 = I_G + M h^2.$$

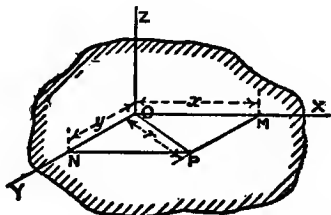
COROLLARY.—Denoting the radii of gyration of the body about the axes, O, and G, by k_0 , and k_G , respectively, we get:—

$$M k_0^2 = M k_G^2 + M h^2.$$

$$\therefore k_0^2 = k_G^2 + h^2. \quad \dots \dots \dots (IV)$$

PROPOSITION II.—If I_x and I_y respectively denote the moments of inertia of a lamina, or plane area, about two axes O X, O Y, at right angles, lying in the plane of the lamina or area, and I_z , the moment of inertia about an axis, O Z, through O, perpendicular to the plane of the lamina or area; then I_z is equal to the sum of I_x , and I_y .

$$i.e., \quad I_z = I_x + I_y. \quad \dots \dots \dots (V)$$



MOMENT OF INERTIA OF LAMINA ABOUT RECTANGULAR AXES.

Take any particle, P, of mass, m , and draw P M, and P N, perpendicular to O X and O Y respectively.

Let x and y denote the co-ordinates of P, with respect to the axes, O X, O Y, so that O M = x , O N = y , and O P = r . Then:—

$$\text{Moment of inertia of P about O X} = m \cdot P M^2 = m \cdot y^2.$$

$$,, \quad ,, \quad O Y = m \cdot P N^2 = m \cdot x^2.$$

$$,, \quad ,, \quad O Z = m \cdot O P^2 = m \cdot r^2.$$

$$\text{But,} \quad r^2 = y^2 + x^2,$$

$$\therefore m \cdot r^2 = m \cdot y^2 + m \cdot x^2.$$

Hence, the moment of inertia of P, about the axis, O Z, is equal to the sum of the moments of inertia of the same particle about the axes, O X, and O Y. But this is equally true for every other particle.

$$\therefore \quad \Sigma m r^2 = \Sigma m y^2 + \Sigma m x^2.$$

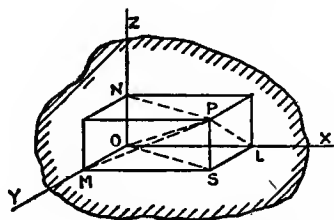
$$\text{i.e.,} \quad I_z = I_x + I_y.$$

COROLLARY.—Denoting the radii of gyration of the lamina, about the axes by the letters k_x, k_y, k_z , we get :—

$$k_z^2 = k_x^2 + k_y^2. \quad \dots \dots \dots \text{(VI)}$$

PROPOSITION III.—If I_x, I_y, I_z respectively denote the moments of inertia of any body about three rectangular axes drawn from any point, O, in the body, then the sum, $I_x + I_y + I_z$, is equal to twice the moment of inertia, I_0 , of the body about the resultant axis which passes through the point, O.

$$\text{i.e.,} \quad I_x + I_y + I_z = 2 I_0 \quad \dots \dots \dots \text{(VII)}$$



MOMENT OF INERTIA OF BODY ABOUT RECTANGULAR AXES.

Let OX, OY, OZ be the three rectangular axes drawn from any point, O; P, any particle of mass m , whose co-ordinates are x, y, z , so that $OL = x, OM = y, ON = z$, and $OP = r$. Then :—

$$\left. \begin{array}{l} \text{Moment of inertia} \\ \text{of } P \text{ about} \end{array} \right\} \begin{array}{l} \text{OX} = m \cdot PL^2 = m(y^2 + z^2). \\ \text{OY} = m \cdot PM^2 = m(z^2 + x^2). \\ \text{OZ} = m \cdot PN^2 = m(x^2 + y^2). \end{array}$$

$$\text{,,} \quad \text{,,} \quad \text{OY} = m \cdot PM^2 = m(z^2 + x^2).$$

$$\text{,,} \quad \text{,,} \quad \text{OZ} = m \cdot PN^2 = m(x^2 + y^2).$$

$$\therefore \quad I_x = \Sigma m(y^2 + z^2) = \Sigma m y^2 + \Sigma m z^2,$$

$$I_y = \Sigma m(z^2 + x^2) = \Sigma m z^2 + \Sigma m x^2,$$

$$\text{And,} \quad I_z = \Sigma m(x^2 + y^2) = \Sigma m x^2 + \Sigma m y^2.$$

$$\therefore \quad I_x + I_y + I_z = 2 \{ \Sigma m x^2 + \Sigma m y^2 + \Sigma m z^2 \}$$

$$\text{,,} \quad \text{,,} \quad = 2 \Sigma m(x^2 + y^2 + z^2),$$

$$\text{i.e.,} \quad I_x + I_y + I_z = 2 \Sigma m r^2 = 2 I_0.$$

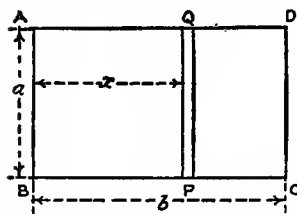
COROLLARY.—Denoting the radii of gyration of the body about the three axes and the point, O, by the letters k_x , k_y , k_z , and k_o respectively, we get :—

$$k_x^2 + k_y^2 + k_z^2 = 2k_o^2. \quad \dots \quad \text{(VIII)}$$

Proposition II. is a particular case of this general one. There O P and O S will always coincide, and, therefore, I_z and I_o are identical. By putting $I_o = I_z$ in equation (VII), we at once get equation (V).

Methods of Calculating Moments of Inertia.—We shall show by working out a few examples how the moments of inertia or radii of gyration can be calculated in certain cases, wherein the density is uniform.

EXAMPLE I.—Determine the moment of inertia and radius of gyration of a rectangular lamina (1) about its shorter edge,



MOMENT OF INERTIA OF
RECTANGLE ABOUT AB.

(2) about an axis in its plane through its *c.g.* and parallel to a short edge, and (3) about an axis through its *c.g.* perpendicular to its plane.

ANSWER.—Let A B C D be the rectangular lamina, and let the edge A B = a , and B C = b .

(1) *About the shorter edge A B.*

Divide the rectangle into n , equal and narrow strips, P Q, parallel to the axis A B.

Let M = Mass of whole figure, A B C D.

„ m = Mass of elementary rectangle, P Q.

„ x = Distance of P Q from axis A B.

„ h = Breadth of elementary strip P Q = $\frac{b}{n}$.

The whole of the strip P Q is at the same distance from A B,

$$\therefore \left. \begin{array}{l} \text{Mom. of inertia of} \\ \text{element P Q} \end{array} \right\} = m x^2.$$

$$\therefore \left. \begin{array}{l} \text{Mom. of inertia of} \\ \text{whole figure} \end{array} \right\} = \Sigma m x^2.$$

$$\text{But} \quad m : M = h : b.$$

$$\therefore \quad m = \frac{M h}{b}.$$

$$\therefore \quad I = \frac{M}{b} \Sigma h x^2. \quad \dots \quad (1)$$

Beginning at edge A B, the distances of the various strips from this edge will be $x_0 = 0$, $x_1 = h$, $x_2 = 2h$, . . . $x_n = nh$.

$$\begin{aligned}\therefore \Sigma h x^2 &= h(0^2 + h^2 + 2^2 h^2 + 3^2 h^2 + \dots + n^2 h^2), \\ &= h^3(1^2 + 2^2 + 3^2 + \dots + n^2), \\ &= h^3 \frac{n(n+1)(2n+1)}{6}, \text{ [See Treatises on Algebra]} \\ &= \frac{(nh)^3}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right).\end{aligned}$$

When the number of strips, n , is infinitely large, the reciprocal, $\frac{1}{n}$, will be infinitely small, and may, therefore, be neglected.

$$\text{Also,} \quad (nh)^3 = b^3.$$

$$\therefore \Sigma h x^2 = \frac{b^3}{6} \times 2 = \frac{1}{3} b^3,$$

$$\therefore \text{From eqn. (1),} \quad I = \frac{1}{3} M b^2.$$

Let k = Radius of gyration, then :—

$$M k^2 = I,$$

$$\therefore k^2 = \frac{1}{3} b^2,$$

$$\text{Or,} \quad k = \frac{b}{\sqrt{3}}.$$

The above method of finding the moment of inertia is precisely the same as that employed in higher mathematics. For those who understand the calculus we here repeat the above calculation, using its notation.

Let dx = Breadth of elementary strip, P Q.

$$\text{Then,} \quad m = \frac{M}{b} dx.$$

$$\therefore dI = \frac{M}{b} x^2 dx.$$

$$\therefore I = \frac{M}{b} \int_0^b x^2 dx = \frac{M}{b} \left[\frac{x^3}{3} \right]_0^b = \frac{M}{b} \cdot \frac{b^3}{3} = \frac{1}{3} M b^2.$$

(2) *About an axis through c.g. parallel to edge A B.*—We may obtain the moment of inertia in this case by proceeding in

exactly the same way as before, but there is no need for this repetition, as we can very easily get the result from the relation given in PROPOSITION I.

Let I_G = Moment of inertia of the rectangle about an axis through its *c.g.* parallel to the edge A B.

„ I = Moment of inertia about the edge A B = $\frac{1}{3} M b^2$.

„ h = Distance between these axes = $\frac{1}{2} b$.

Then, from equation (III):—

$$I = I_G + M h^2.$$

$$\therefore I_G = I - M h^2 = \frac{1}{3} M b^2 - \frac{1}{4} M b^2 = \frac{1}{12} M b^2.$$

$$\text{Also, } k_G^2 = \frac{1}{12} b^2.$$

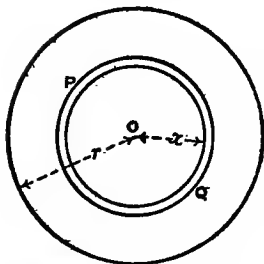
(3) *About an axis through c.g. perpendicular to plane.*

$$\left. \begin{array}{l} \text{Moment of inertia about an axis} \\ \text{through c.g. parallel to A B} \end{array} \right\} = I_x = \frac{1}{12} M b^2.$$

$$\left. \begin{array}{l} \text{Moment of inertia about an axis} \\ \text{through c.g. parallel to B C} \end{array} \right\} = I_y = \frac{1}{12} M a^2$$

$$\text{But, from equation (V), } I_z = I_x + I_y.$$

$$\therefore \left. \begin{array}{l} \text{Moment of inertia about an axis} \\ \text{through c.g. perpendicular to} \\ \text{the plane} \end{array} \right\} = I_z = \frac{1}{12} M (a^2 + b^2).$$



RADIUS OF GYRATION OF A CIRCULAR DISC.

EXAMPLE II.—Determine the radius of gyration of a circular disc about an axis through its centre perpendicular to its plane.

ANSWER.—Divide the disc into n , equal, narrow rings of breadth, h . Taking one of these rings, P Q, let x be its distance from the centre, O.

Let M = Mass of the disc.

„ m = Mass of the elementary ring, P Q.

„ r = Radius of the disc.

Then, $m : M = \text{area of ring} : \text{area of disc}.$

$$\text{i.e., } m : M = 2 \pi x h : \pi r^2.$$

$$\therefore m = \frac{2 M}{r^2} x h$$

The whole of the elementary ring is at the same distance from the centre :—

$$\therefore \left. \begin{array}{l} \text{Moment of inertia of elementary} \\ \text{ring about the axis through O} \end{array} \right\} = m x^2 = \frac{2 M}{r^2} x^3 h.$$

$$\therefore \left. \begin{array}{l} \text{Moment of inertia of the whole} \\ \text{disc} \end{array} \right\} = I = \frac{2 M}{r^2} \Sigma x^3 h. \quad (1)$$

Beginning at the centre, O, the distances of the various rings will be $x_0 = 0$, $x_1 = h$, $x_2 = 2h$, . . . $x_n = nh$.

$$\begin{aligned} \therefore \Sigma x^3 h &= (0^3 + 1^3 h^3 + 2^3 h^3 + \dots + n^3 h^3) h \\ &= (1^3 + 2^3 + 3^3 + \dots + n^3) h^4 \\ &= \frac{1}{4} n^2 (n + 1)^2 h^4 \quad [\text{See Treatises on Algebra}] \\ &= \frac{(nh)^4}{4} \cdot \left(1 + \frac{1}{n}\right)^2. \end{aligned}$$

When n is infinitely great, the reciprocal, $\frac{1}{n}$, will be infinitely small, and may be neglected.

$$\text{Also,} \quad (nh)^4 = r^4.$$

$$\therefore \text{From equation (1), } I = \frac{2 M}{r^2} \times \frac{r^4}{4} = \frac{1}{2} M r^2.$$

$$\therefore \quad k^2 = \frac{1}{2} r^2. \quad \dots \dots \dots (2)$$

These results may also be obtained by the aid of the calculus, thus :—

Let dx = Breadth of elementary ring.

$$\text{Then,} \quad m = \frac{2 M}{r^2} x dx.$$

$$\therefore \quad dI = \frac{2 M}{r^2} x^3 dx.$$

$$\therefore \quad I = \frac{2 M}{r^2} \int_0^r x^3 dx = \frac{2 M}{r^2} \cdot \frac{r^4}{4} = \frac{1}{2} M r^2.$$

If, however, the disc be annular, the outside and inside radii being R and r respectively, we get :—

$$m = \frac{2 M}{R^2 - r^2} \cdot x dx.$$

$$\begin{aligned}\text{And, } I &= \frac{2M}{R^2 - r^2} \int_r^R x^3 dx \\ , , &= \frac{2M}{R^2 - r^2} \times \frac{R^4 - r^4}{4} = \frac{1}{2} M (R^2 + r^2). \\ \therefore k^2 &= \frac{1}{2} (R^2 + r^2).\end{aligned}$$

These results also express the moments of inertia and radii of gyration of a solid and of a hollow cylinder about their axes. For a cylinder can be conceived as made up of a great number of circular discs threaded together on the same axis, and the moment of inertia will just be the sum of the moments of inertia of all the discs, since the radius of gyration of each disc is independent of the thickness of the disc, it follows that the radius of gyration of the whole cylinder will be the same as that of one of the discs.

Having found the radius of gyration of a circular disc about an axis through its centre at right angles to its plane, we can very easily find its radius of gyration about a diameter.

Let k_x, k_y = Radii of gyration of disc about two diameters at right angles to each other.

, , k_z = Radius of gyration about axis through centre and perpendicular to plane.

Then, $k_x = k_y = k$

And, from (2) $k_z^2 = \frac{1}{2} r^2$.

But, from equation (VI), PROPOSITION II., we get:—

$$k_x^2 + k_y^2 = k_z^2$$

$$\therefore 2k^2 = k_z^2 = \frac{1}{2} r^2.$$

$$\therefore k^2 = \frac{1}{4} r^2, \text{ or } k = \frac{r}{2}.$$

If the disc be annular and of radii R and r , then the radius of gyration about any diameter, is given by the equation:—

$$k^2 = \frac{1}{4} (R^2 + r^2).$$

EXAMPLE III.—Determine the radius of gyration of a sphere about a diameter.

ANSWER.—The results of PROPOSITION III. tell us that if three mutually perpendicular axes be drawn from any point

in a body, the sum of the moments of inertia of the body about these axes is equal to twice the moment of inertia of the body about that point. Suppose, then, that the point selected be the centre of the sphere, the axes will then be three mutually perpendicular diameters. But the moments of inertia about all diameters must be the same. Therefore, if I denote the moment of inertia of the sphere about any diameter and I_0 that about the centre, O , we get, from equation (VII):—

$$3 I = 2 I_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It only remains now to find the value of I_0 or $\Sigma m x^2$.

Suppose the sphere divided into a large number n , of concentric shells, the thickness of each shell being h .

Let x = Distance of any one shell from centre of sphere.

„ r = Radius of sphere.

„ m = Mass of shell.

„ M = Mass of sphere.

Then, $m : M = \text{vol. of shell} : \text{vol. of sphere},$

$$\text{i.e.,} \quad m : M = 4 \pi x^2 h : \frac{4}{3} \pi r^3,$$

$$\therefore \quad m = \frac{3 M}{r^3} x^2 h,$$

$$\text{And,} \quad I_0 = \Sigma m x^2 = \frac{3 M}{r^3} \Sigma x^4 h.$$

Beginning at the centre of the sphere and putting successively, $x_0 = 0, x_1 = h, x_2 = 2h, \dots, x_n = nh$, we get:—

$$\Sigma x^4 h = (1^4 + 2^4 + 3^4 + \dots + n^4)h^5,$$

$$,, = \left\{ \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} \right\} h^5,$$

$$,, = \left\{ \frac{1}{5} + \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{30n^4} \right\} n^5 h^5.$$

When n is infinitely great, the quantity inside the brackets reduces to $\frac{1}{5}$, and

$$n^5 h^5 = r^5.$$

$$\therefore \quad \Sigma x^4 h = \frac{1}{5} r^5.$$

$$\therefore \quad I_0 = \frac{3 M}{r^3} \times \frac{1}{5} r^5 = \frac{3}{5} M r^2.$$

∴ From equation (I) $I = \frac{2}{3} I_0 = \frac{2}{5} M r^2$.

Hence, $k^2 = \frac{2}{5} r^2$.

The same result can be easily arrived at by aid of the Calculus. With the usual notation, we get :—

$$m = \frac{3 M}{r^3} x^2 dx,$$

$$\therefore I_0 = \sum m x^2 = \frac{3 M}{r^3} \int_0^r x^4 dx = \frac{3 M}{r^3} \times \frac{r^5}{5} = \frac{3}{5} M r^2,$$

$$\therefore I = \frac{2}{3} I_0 = \frac{2}{5} M r^2.$$

If the sphere be hollow, the inside radius being r , and the outside radius R , it can easily be proved that :—

$$I = \frac{2}{5} M \frac{R^5 - r^5}{R^3 - r^3}.$$

The term “moment of inertia” has been defined above with respect to a solid body only, but it is easy to see that by a slight alteration in the wording of the definition it may be made to apply equally to an area or a section of a solid. Accordingly, we find the terms “moment of inertia” and “radius of gyration” applied to areas as well as to solids. Thus, we speak about the moment of inertia and radius of gyration of a circle about a diameter, a triangle about its base, and so on.

We may here remark that the moment of inertia of a solid, or section of a solid, about a given axis, is always proportional to the mass of the solid, or to the area of the section as the case may be.

The following rule has been stated by Routh and will be found useful for finding the moments of inertia about an axis of symmetry :—

Moment of Inertia = Mass × (sum of the squares of the perpendicular semi-axes) ÷ (3, 4, or 5, according as the body is rectangular, elliptical, or ellipsoidal).

For the sake of reference, we here give tables of the squares of the radii of gyration for some of the more important cases of both solids and sections.

In every case the axis is taken as passing through the centre of mass of the solid or centre of area of the section, so that if the moment of inertia or radius of gyration be required about any other axis, this can easily be computed from the results given in PROPOSITIONS I., II., and III.

TABLE I.—SQUARES OF RADII OF GYRATION OF SOLIDS.

	Name of Solid, and Dimensions.	Position of Axis through c.g.	Square of Radius of Gyration. $k^2 = \frac{I}{M}$
I.	Circular hoop of thin wire —Radius, r	Perp. to plane of circle	r^2
II.	Circular hoop of thin wire —Radius, r	About a diameter	$\frac{1}{2} r^2$
III.	Uniform circular rod — Length, l ; radius, r	Perp. to length	$\frac{1}{12} l^2 + \frac{1}{4} r^2$
IV.	Solid circular cylinder — Radius, r	About its own axis	$\frac{1}{2} r^2$
V.	Hollow circular cylinder or ring—Radii, R, r	About its own axis	$\frac{1}{2} (R^2 + r^2)$
VI.	Thin cylindrical shell — Radius, r	About its own axis	r^2
VII.	Solid sphere—Radius, r	About a diameter	$\frac{2}{5} r^2$
VIII.	Hollow sphere—Radii, R, r	About a diameter	$\frac{2}{5} \frac{R^5 - r^5}{R^3 - r^3}$
IX.	Thin spherical shell — Radius, r	About a diameter	$\frac{2}{3} r^2$
X.	Solid cone—Radius of base, r	About its own axis	$\frac{3}{10} r^2$

TABLE II.—SQUARES OF RADII OF GYRATION OF LAMINA AND SURFACES OR SECTIONS.

	Form of Lamina, Surface, or Section.	Position of Axis through c.g.	Square of Radius of Gyration. $k^2 = \frac{I}{A}$
I.	Rectangle—Sides, a, b	Parallel to side, b	$\frac{1}{12} a^2$
II.	Rectangle—Sides, a, b	Perp. to plane of figure	$\frac{1}{12} (a^2 + b^2)$
III.	Hollow rectangle — Sides, A, B , and a, b	Parallel to sides, B, b	$\frac{1}{12} \frac{A^3 B - a^3 b}{A B - a b}$
IV.	Triangle — Altitude, a ; base, b	Parallel to base, b	$\frac{1}{18} a^2$
V.	Circular section—Radius, r	Perp. to plane of figure	$\frac{1}{2} r^2$
VI.	Circular section—Radius, r	About a diameter	$\frac{1}{4} r^2$
VII.	Hollow circular section— Radii, R, r	Perp. to plane of figure	$\frac{1}{2} (R^2 + r^2)$
VIII.	Hollow circular section— Radii, R, r	About a diameter	$\frac{1}{4} (R^2 + r^2)$
IX.	Elliptical section—Axes, a, b	About axis, b	$\frac{1}{16} a^2$
X.	Elliptical section—Axes, a, b	Perp. to plane of figure	$\frac{1}{16} (a^2 + b^2)$
XI.	Hollow elliptical section— Axes, A, B , and a, b	About axis, B, b	$\frac{1}{16} \frac{A^3 B - a^3 b}{A B - a b}$

Equation of Energy for a Rotating Body.—We shall now determine the energy possessed by a rotating body.

Let W = Weight of body, and M its mass.

„ w = Weight of any particle at a distance, r , from the axis of rotation, and m its mass.

„ ω = Angular velocity of body about given axis.

„ v = Linear velocity of the particle = ωr .

„ k = Radius of gyration about the given axis.

$$\text{Then, the kinetic energy of the particle} = \frac{w v^2}{2g} = \frac{w \omega^2 r^2}{2g}.$$

Repeating this process for every particle composing the body, and adding the results together, we get:—

$$\left. \begin{array}{l} \text{The kinetic energy of} \\ \text{the whole body, } E_k \end{array} \right\} = \sum \frac{w \omega^2 r^2}{2g} = \frac{\omega^2}{2g} \sum w r^2 = \frac{\omega^2}{2} \sum m r^2,$$

since $w = m g$, and ω is the same for every particle.

$$\text{But,} \quad \sum m r^2 = I = M k^2 = \frac{W k^2}{g}, \text{ about the given axis,}$$

$$\therefore \quad E_k = \frac{1}{2} I \omega^2 = \frac{W \omega^2 k^2}{2g} \quad \dots \dots (IX)^*$$

Thus, the equation for the energy of a body rotating about a fixed axis is similar in form to that for a body moving without rotation.

Engineers usually measure the angular velocity, ω , of a rotating body by the number of revolutions made in unit time.

Then, if n be the number of revolutions per unit time,

$$\omega = 2 \pi n$$

$$\therefore \quad E_k = \frac{W \times 4 \pi^2 n^2 k^2}{2g} = \frac{2 \pi^2 n^2 W k^2}{g} \quad \dots \dots (X)$$

We may also show, as in the previous Lecture, that, if the angular velocity changes from ω_1 to ω_2 , or from n_1 to n_2 revolutions per second, then:—

* If W be expressed in absolute units or pounds, the kinetic energy will also be given in absolute units or foot-pounds; but if W be in pounds weight or in gravitation units, then the kinetic energy will be in foot-pounds.

The student should note that the *pound* is the absolute unit of mass, and, therefore, those of the above equations which contain M instead of $\frac{W}{g}$ *always* give the kinetic energy in absolute units or foot-pounds.

$$\left. \begin{array}{l} \text{The change of kinetic energy} \\ \text{Or, " " } \end{array} \right\} = \frac{W(\omega_2^2 - \omega_1^2)k^2}{2g} = \frac{2\pi^2 W(n_2^2 - n_1^2)k^2}{g} \quad (\text{XI})$$

Again, if the centre of gravity of the body be moving with a linear velocity, v , and if at the same time the body be rotating about an axis through its centre of gravity with an angular velocity, ω , then the total kinetic energy possessed by the body is :—

$$\mathbf{E}_k = \frac{W v^2}{2a} + \frac{W \omega^2 k^2}{2a} = \frac{W}{2a} \{v^2 + \omega^2 k^2\}. \quad (\text{XII})$$

Or, if the linear velocity changes from v_1 to v_2 , while the angular velocity changes from ω_1 to ω_2 , then the total change in the kinetic energy of the body during that period is :—

$$\frac{W(v_2^2 - v_1^2)}{2g} + \frac{W(\omega_2^2 - \omega_1^2)/k^2}{2g} = \frac{W}{2g} \{ (v_2^2 - v_1^2) + (\omega_2^2 - \omega_1^2)k^2 \}. \quad (\text{XIII})$$

EXAMPLE IV.—Sketch and describe the action of a fly-press as used for punching holes in metal plates. The balls weigh 80 lbs. each, and are fixed at a radius of 30 inches from the axis of the screw. The screw is double threaded, and of 1 inch pitch. Find what diameter of hole can be punched in a wrought iron plate $\frac{3}{8}$ inch thick, if the strength of the plate in shear be taken at 22 tons per square inch, the resistance to shearing be overcome in the first $\frac{1}{16}$ inch, and if the balls at the instant when the punch touches the plate are moving at the rate of 60 revolutions per minute.

ANSWER.—For a sketch and description of a fly-press, the student may refer to Lecture XXI., of the Author's *Elementary Manual on Applied Mechanics*.

Let W = Weight of *each* ball = 60 lbs.

„ k = Radius of gyration of the system = $2\frac{1}{2}$ feet.

„ n = Number of revolutions *per second* = 1.

„ R = Resistance, in lbs., offered by the metal to the punch.

„ $s = \text{Distance through which R is overcome} = \frac{1}{16} \text{ inch}$
 $= \frac{1}{12 \times 16} \text{ feet.}$

„ t = Thickness of plate punched = $\frac{3}{8}$ inch.

.. d = Diameter, in inches, of the hole.

„ $f =$ Resistance of metal to shearing $= 22 \times 2240$ lbs.
per square inch.

Then, $\text{Area sheared} = \left\{ \begin{array}{l} \text{Area of cylindrical surface of} \\ \text{hole} = \pi d t. \end{array} \right.$

$\therefore \text{Mean resistance offered to shearing} \} = R = \pi d t f.$

$\therefore \text{Work done against } R = R \times s = \pi d t f \times s.$

But, $\text{Work done against } R = \left\{ \begin{array}{l} \text{Energy of moving balls at the} \\ \text{instant when punch strikes} \\ \text{metal} \end{array} \right.$

$$= \frac{2 W v^2}{2 g} = \frac{W \times 4 \pi^2 n^2 k^2}{g}.$$

$\therefore \pi d t f \times s = \frac{W \times 4 \pi^2 n^2 k^2}{g}.$

This is the general equation connecting together the given and the required quantities. By substituting the given data, and cancelling π from both sides of the equation, we get :—

$$d \times \frac{8}{8} \times 22 \times 2240 \times \frac{1}{12 \times 16} = \frac{60 \times 4 \times \frac{22}{7} \times 1 \times 1 \times 2\frac{1}{2} \times 2\frac{1}{2}}{32}$$

$\therefore d = 1.53 \text{ inch.}$

EXAMPLE V.—A flywheel weighing 4 tons is keyed to a shaft of 9 inches diameter at the journals. The radius of gyration of the wheel is $5\frac{1}{4}$ feet. At a given instant the wheel is found to be making 80 revolutions per minute, and is not acted on by any other retarding forces than the friction at its journals. Find (1) the reduction in speed after the wheel has made 100 turns. (2) The number of turns it will make before it stops if the coeff. of friction between the journals and their bearings = 0.07.

ANSWER.—(1) To find the reduction in speed after the wheel has made 100 turns, we must equate the work done against friction in 100 turns to the change of kinetic energy of the wheel during that time.

Let W = Weight of wheel = 4 tons = $4 \times 2,240$ lbs.

„ k = Radius of gyration of wheel = $5\frac{1}{4}$ ft.

„ n_1, n_2 = Initial and final revolutions per second.

„ d = Diameter of journals = $\frac{3}{4}$ ft.

„ μ = Coefficient of friction = 0.07.

Using equation (XI), we have :—

$$\text{Change of } E_k \text{ of wheel} = \frac{2 \pi^2 (n_1^2 - n_2^2) W k^2}{g},$$

And, from equation (II_b), Lecture VII., Vol. I. :—

Work lost in friction in one turn of journals = $\pi d \mu R$.

Where R is the resultant pressure on the bearings, and therefore = W in this case.

$$\therefore \left. \begin{array}{l} \text{Work done against friction} \\ \text{in 100 turns} \end{array} \right\} = 100 \pi d \mu W.$$

$$\therefore \frac{2 \pi^2 (n_1^2 - n_2^2) W k^2}{g} = 100 \pi d \mu W.$$

$$\therefore n_1^2 - n_2^2 = \frac{100 d \mu g}{2 \pi k^2}.$$

$$\text{Hence,} \quad n_2^2 = n_1^2 - \frac{50 d \mu g}{\pi k^2}$$

$$\therefore n = \left(\frac{80}{60}\right)^2 - \frac{50 \times .75 \times .07 \times 32}{\frac{22}{7} \times 5.25 \times 5.25}$$

$$,, = 1.78 - .97 = .81.$$

$$\text{Or,} \quad n_2 = \sqrt{.81} = .9 \text{ rev. per sec., or } 54 \text{ revs. per min.}$$

$$\therefore \text{Reduction in speed} = n_1 - n_2 = 80 - 54 = 26 \text{ revs. per min.}$$

(2) Let n = number of turns made before stopping.

Then, in this case, the whole energy of the wheel when making 80 turns per minute is absorbed in friction at the journals.

$$\therefore \frac{2 \pi^2 n_1^2 W k^2}{g} = n \pi d \mu W.$$

$$\therefore n = \frac{2 \pi n_1^2 k^2}{\mu d g} = \frac{2 \times \frac{22}{7} \times \frac{80}{60} \times \frac{80}{60} \times 5.25 \times 5.25}{.07 \times .75 \times 32}$$

$$\therefore n = 183\frac{1}{3} \text{ turns.}$$

EXAMPLE VI.—A right cylinder of radius r , rolls, without slipping, down an inclined plane of height h . Find its velocity at the foot of the plane, and compare this with that which it

would have had by merely sliding. Neglect frictional resistances in both cases.

ANSWER.—Let v = Velocity of *c.g.* of cylinder at foot of plane.

„ ω = Angular velocity „ „

„ W = Weight of cylinder.

„ k = Radius of gyration about its own axis

$$= \frac{r}{\sqrt{2}}.$$

Then, $\left. \begin{array}{l} \text{Total kinetic energy} \\ \text{at foot of plane} \end{array} \right\} = \left\{ \begin{array}{l} \text{Energy of Translation} \\ + \text{Energy of Rotation.} \end{array} \right.$

But, $\left. \begin{array}{l} \text{Total energy at foot} \\ \text{of plane} \end{array} \right\} = Wh.$

Also, $\text{Energy of Translation} = \frac{W v^2}{2g}.$

And, $\text{Energy of Rotation} = \frac{W \omega^2 k^2}{2g}.$

$$\therefore Wh = \frac{W v^2}{2g} + \frac{W \omega^2 k^2}{2g}. \quad \therefore 2gh = v^2 + \omega^2 k^2.$$

$$\text{But, } \omega = \frac{v}{r}, \text{ and } k = \frac{r}{\sqrt{2}}. \quad \therefore \omega^2 k^2 = \frac{v^2}{2}.$$

$$\therefore 2gh = v^2 + \frac{v^2}{2}. \quad \therefore v = \sqrt{\frac{4gh}{3}}.$$

Had the cylinder been allowed to *slide* down the plane *without rolling*, the velocity at foot of plane would have been:—

$$v = \sqrt{2gh}.$$

$$\therefore \left. \begin{array}{l} \text{Vel. with rolling} : \text{Vel.} \\ \text{without rolling} \end{array} \right\} = \sqrt{\frac{4gh}{3}} : \sqrt{2gh} = \sqrt{2} : \sqrt{3}.$$

Of course, the kinetic energy of the body in both cases is the same, but in the second case the whole energy is translational, hence the reason for the greater speed in this case.

EXAMPLE VII.—A weight, Q , draws up another weight, W , by means of an ordinary wheel and axle. The force ratio ($Q : W$) is 1 to 6, and the velocity ratio (vel. of Q : vel. of W) is 8 to 1. The diameter of the axle is 6 inches, and the radius

of gyration of the wheel and its axle may be taken at 10 inches. Neglecting frictional resistances and the inertia of the ropes, determine the revolutions per minute of the machine after 10 turns have been made from a state of rest. Take the weight of the wheel and its axle = $2W$.

ANSWER.—We shall first answer this question in a general way.

Let W_1 = Weight of wheel and axle.

„ V = Velocity of effort, Q , in ft. per sec., after N turns.

„ v = Velocity of weight, W , „ „

„ R = Radius of wheel in feet.

„ r = Radius of axle „

„ k = Radius of gyration of wheel and axle in feet.

„ n = Revolutions per sec. of machine, after N turns.

Then, by the *Principle of Energy*, we get:—

Energy exerted = Work done + Change of kinetic energy.

But, $\left. \begin{array}{l} \text{Energy} \\ \text{exerted} \end{array} \right\} = Q \times \text{Distance fallen in } N \text{ turns of machine.}$

$$,, = Q \times 2 \pi R N.$$

Work done = $\left\{ \begin{array}{l} W \times \text{Distance raised in } N \text{ turns of} \\ \text{machine} = W \times 2 \pi r N. \end{array} \right.$

Change of kinetic $\left\{ = \left\{ \begin{array}{l} \text{Translational energy of } Q \text{ and } W + \text{Rota-} \\ \text{energy} \quad \quad \quad \text{tional energy of wheel and axle.} \end{array} \right. \right.$

$$,, = \frac{4\pi^2 n^2 Q \times R^2}{2g} + \frac{4\pi^2 n^2 W \times r^2}{2g} + \frac{4\pi^2 n^2 W_1 \times k^2}{2g}$$

$$,, = \frac{2\pi^2 n^2}{g} \left\{ Q R^2 + W r^2 + W_1 k^2 \right\}$$

$$\text{Hence, } Q \times 2\pi R N = W \times 2\pi r N + \frac{2\pi^2 n^2}{g} \left\{ Q R^2 + W r^2 + W_1 k^2 \right\}$$

$$\text{Or, } (Q R - W r) N = \frac{\pi n^2}{g} \left\{ Q R^2 + W r^2 + W_1 k^2 \right\}$$

This is the general expression from which n can be found when the other quantities are given.

From the question, we get:— $W = 6Q$; $W_1 = 2W = 12Q$;
 $N = 10$; $V = 8v$; $r = 3 \text{ inches} = \frac{1}{4} \text{ foot}$; $R = \frac{V}{v} r = 8 \times \frac{1}{4}$
 $= 2 \text{ feet}$; $k = \frac{10}{12} = \frac{5}{6} \text{ foot.}$

$$\text{Hence, } \left. \begin{array}{l} \text{Energy} \\ \text{exerted} \end{array} \right\} = Q \times 2\pi RN = Q \times 2\pi \times 2 \times 10 = 40\pi Q \text{ ft.-lbs.}$$

$$\text{Work done} = W \times 2\pi rN = 6Q \times 2\pi \times \frac{1}{2} \times 10 = 30\pi Q \quad ,,$$

$$\left. \begin{array}{l} \text{Change of kinetic} \\ \text{energy} \end{array} \right\} = \frac{2\pi^2 n^2}{g} \left\{ Q R^2 + W r^2 + W_1 k^2 \right\}$$

$$,, \quad ,, \quad = \frac{2\pi^2 n^2}{32} \left\{ Q \times 2^2 + 6Q \times \left(\frac{1}{2}\right)^2 + 12Q \times \left(\frac{5}{8}\right)^2 \right\} ,,$$

$$,, \quad ,, \quad = \frac{\pi^2 n^2}{16} \times \frac{305}{24} Q \text{ ft.-lbs.}$$

$$\therefore \quad 40\pi Q = 30\pi Q + \frac{305\pi^2 n^2}{16 \times 24} Q$$

$$\therefore \quad n^2 = \frac{10 \times 16 \times 24}{305 \times \frac{22}{7}} = 4.$$

$$\therefore \quad n = 2 \text{ revolutions per second, or } 120 \text{ r.p.m.}$$

Determination of the Energy of Flywheels.—Before the energy of a rotating body can be calculated at any given speed, it is necessary to know the radius of gyration of that body about the given axis of rotation. We have already shown how this quantity can be calculated in certain bodies which are of regular geometrical form; but many cases occur in the rotating parts of machines where the above methods of calculation would be most difficult, if not altogether impossible. Such is the case with most flywheels. The flywheel is a most important part of an engine, since it is a regulator of the speed. Owing to the great mass of its rim it naturally possesses great inertia, and is, therefore, capable of storing up a considerable amount of the energy developed in the cylinder, and of again imparting this stored energy to the moving parts during those portions of a revolution when the work done in the cylinder is less than the work being done outside. It is important to know the radius of gyration of the wheel, so that calculations relating to the storage and output of its energy can be effected. This radius of the wheel may be determined either approximately by calculation, or accurately by experimenting on the wheel itself, or with another similarly shaped wheel. We shall deal with these cases in turn.

(1) *By Approximate Calculation.*—Most flywheels consist of a heavy rim with comparatively light arms and nave; hence,

in calculations relating to the radii of gyration of such wheels, we may neglect the effects of the arms and nave, and consider only that of the heavy rim. Usually the rim is of a rectangular cross section.

Let R, r = Outside and inside radii of rim.

Then, from Table I., case V., of this Lecture, we get :—

$$k^2 = \frac{1}{2} (R^2 + r^2).$$

Substituting this in the equation for the kinetic energy of the wheel, we may obtain an approximate result.

Many engineers, however, further simplify their formula by taking for the radius of gyration the *mean* radius of the rim, and consider this quite near enough for most purposes. Thus:—

$$k = \frac{1}{2} (R + r).$$

The difference in the kinetic energy, as calculated from those two assumptions, may be shown as follows :—

Let W = Total weight of wheel.

„ ω = Angular velocity of wheel.

Then, according to the *first* assumption :—

$$\text{The kinetic energy} = \frac{W \omega^2 k^2}{2g} = \frac{W \omega^2}{2g} \times \frac{R^2 + r^2}{2}.$$

And, according to the *second* assumption :—

$$\text{The kinetic energy} = \frac{W \omega^2}{2g} \times \frac{(R + r)^2}{4}.$$

$$\text{Hence, the difference} = \frac{W \omega^2}{2g} \left\{ \frac{R^2 + r^2}{2} - \frac{(R + r)^2}{4} \right\}$$

$$\text{„ „} = \frac{W \omega^2}{2g} \times \frac{(R - r)^2}{4}.$$

That is, the kinetic energy in the *first* case is greater than that in the *second* case by $\frac{W \omega^2}{2g} \times \frac{(R - r)^2}{4}$. This difference, however, becomes less as $R - r$ diminishes—that is, as r approaches R . On the other hand, it gets greater the thicker the rim. The radius of gyration in the first case—viz., $k^2 = \frac{1}{2} (R^2 + r^2)$, is too great; because the effect of the arms

and nave is to reduce that radius, whereas the other result, $k = \frac{1}{2}(R + r)$, may be too small. It sometimes happens that a closer approximation may be obtained by taking the arithmetical *mean* of the above results, thus:—

$$\begin{aligned} \text{The kinetic energy of the wheel} \} &= \frac{W \omega^2}{2g} \times \frac{1}{2} \left\{ \frac{R^2 + r^2}{2} + \frac{(R + r)^2}{4} \right\} \\ \text{" " } &= \frac{W \omega^2}{2g} \times \frac{3(R + r)^2 - 4Rr}{8}. \quad (\text{XIV}) \end{aligned}$$

(2) *By Experiment on the Wheel.*—When accurate results are required, we may determine the radius of gyration of the wheel experimentally as follows:—

Disconnect the flywheel and its shaft from all other moving pieces, and see that the shaft runs smoothly in its bearings. Fit a flat pulley on the shaft and wind a few turns of flexible rope in a single layer round the same.* To the free end of this rope attach a weight sufficiently heavy to cause the flywheel to rotate at a uniform speed when started by the hand. This weight should just supply the energy absorbed by the friction of the shaft in its bearings and the bending of the rope. Now rewind the rope on the pulley and add another weight to its free end, so that the wheel will now start rotating when the weights are allowed to fall. Note the time taken by the weights in falling a known distance. The height through which the weights fall, and the diameter of the pulley being known, it is easy to calculate both the speed of the wheel and the falling weights, and hence their kinetic energies at the instant when the latter reach the ground.

Another method of allowing for the friction of the bearings, &c., is to use only one weight. Note the exact number of turns which the wheel makes (after the weight has ceased to act) until it comes to rest. Then neglecting the atmospheric resistance (which will be very small in an experiment of this kind) the work absorbed at the bearings will be equal to the kinetic energy of the wheel at the instant when the weight ceases to act.

These methods will be better understood when stated thus:—

* If the flywheel shaft be of sufficient diameter, this pulley may be dispensed with, and the rope need then be simply wound round the shaft. If a convenient direct drop for the weights cannot be arranged for, then the rope may pass round a guide pulley fixed to the roof, but in this case the kinetic energy of this pulley must be allowed for.

Let W = Weight of flywheel in *lbs.*

„ w = Weight producing motion of wheel.

„ w_1 = Weight required to balance friction.

„ k = Radius of gyration of flywheel in *feet.*

„ h = Height through which w and w_1 fall in *feet.*

„ D = Diameter of pulley keyed to shaft in *feet.*

„ d = Diameter of rope in *feet.*

„ t = Time taken by weight, w , in falling to the ground in *seconds.*

„ n = Number of revolutions *per second* which wheel is making at instant when w reaches the ground.

„ v = Velocity with which w and w_1 strike the ground.

„ N = Number of revolutions made by wheel after w ceases to act.

FIRSTLY. — When w_1 is employed to balance the frictional resistances. All the energy exerted by w is employed in giving kinetic energy to the wheel.

But, *Energy exerted* = wh .

And, $\left. \begin{array}{l} \text{Change} \\ \text{of kinetic} \\ \text{energy} \end{array} \right\} = \left\{ \begin{array}{l} \text{Kinetic} \\ \text{energy of} \\ \text{wheel} \end{array} \right\} + \left\{ \begin{array}{l} \text{Kinetic energy of } w \text{ and} \\ w_1 \text{ when they reach} \\ \text{the ground} \end{array} \right\}$

$$= \frac{W \times 4 \pi^2 n^2 k^2}{2g} + \frac{(w + w_1) v^2}{2g}.$$

$$\therefore wh = \frac{W \times 4 \pi^2 n^2 k^2}{2g} + \frac{(w + w_1) v^2}{2g} \quad \dots (1)$$

The revolutions, n , and the linear velocity, v , of the falling body at the instant when the latter reaches the ground can be determined as follows, when t , D , and d are known :—

$$\left. \begin{array}{l} \text{Number of revols. made by} \\ \text{wheel during action of } w \end{array} \right\} = \frac{h}{\pi(D + d)} \quad \dots \dots \dots (2)$$

$$\therefore \left. \begin{array}{l} \text{Average number of} \\ \text{revols. per second} \end{array} \right\} = \frac{h}{\pi(D + d)t}$$

$$\therefore n = \text{Twice the average,}$$

$$= \frac{2h}{\pi(D + d)t} \quad \dots \dots \dots (3)$$

Similarly $v = \text{Twice average linear velocity of } w \text{ and } w_1,$

$$= \frac{2h}{t} \quad \dots \dots \dots (4)$$

∴ From equation (1) we get :—

$$w h = \frac{W \times 4 \pi^2 \times 4 h^2 \times k^2}{2 g \times \pi^2 (D + d)^2 t^2} + \frac{(w + w_1) \times 4 h^2}{2 g \times t^2}$$

$$\therefore \frac{8 W h k^2}{g (D + d)^2 t^2} = w - \frac{2 (w + w_1) h}{g t^2} \quad \dots \quad (\text{XV})$$

SECONDLY.—When the number of turns made by wheel after w ceases to act is known,

$$\text{Energy exerted} = w h.$$

$$\left. \begin{array}{l} \text{Work done on friction dur-} \\ \text{ing last } N \text{ revolutions of} \\ \text{wheel} \end{array} \right\} = \left\{ \begin{array}{l} \text{Kinetic energy of wheel at} \\ \text{instant when } w \text{ ceases to} \\ \text{act} \end{array} \right.$$

$$\begin{array}{ccc} " & " & = \frac{W \times 4 \pi^2 n^2 k^2}{2 g} \end{array}$$

But, by equation (2), the wheel makes $\frac{h}{\pi (D + d)}$ revolutions during the action of w .

$$\therefore \left. \begin{array}{l} \text{Work done on friction} \\ \text{during action of } w \end{array} \right\} = \frac{W \times 4 \pi^2 n^2 k^2}{2 g} \times \frac{\frac{h}{\pi (D + d)}}{N}$$

From equation (3), we get :—

$$n = \frac{2 h}{\pi (D + d) t}$$

$$\therefore \text{Work done} = \frac{W \times 4 \pi^2 \times \frac{4 h^2}{\pi^2 (D + d)^2 t^2} \times k^2}{2 g} \times \frac{h}{\pi (D + d) N}$$

$$" = \frac{8 W h^3 k^2}{g \pi (D + d)^3 t^2 N}$$

$$\left. \begin{array}{l} \text{Change of} \\ \text{kinetic energy} \end{array} \right\} = \frac{W \times 4 \pi^2 n^2 k^2}{2 g} + \frac{w v^2}{2 g}$$

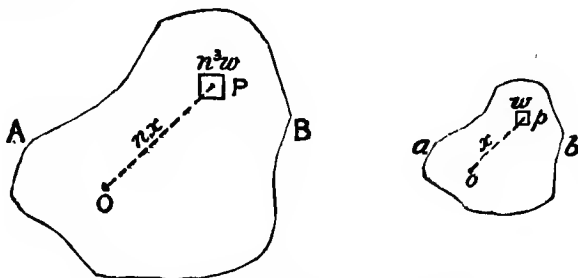
$$" = \frac{8 W h^2 k^2}{g (D + d)^2 t^2} + \frac{2 w h^2}{g t^2}$$

$$\text{Hence, } w h = \frac{8 W h^3 k^2}{g \pi (D + d)^3 t^2 N} + \frac{8 W h^2 k^2}{g (D + d)^2 t^2} + \frac{2 w h^2}{g t^2}$$

$$\therefore \frac{8 W h k^2}{g (D + d)^2 t^2} \left\{ \frac{h}{\pi (D + d) N} + 1 \right\} = w \left\{ 1 - \frac{2 h}{g t^2} \right\} \quad (\text{XVI})$$

Equations (XV) and (XVI) enable us to find the radius of gyration, k , when the data are furnished by experiment.

If the wheel whose radius of gyration has to be found cannot be conveniently experimented on, then the radius of gyration of another similar wheel may be determined, and that for the first wheel calculated therefrom. It is easy to show generally that the moments of inertia of two similar bodies rotating about similarly placed axes are as the fifth powers of their like linear dimensions.



MOMENTS OF INERTIA OF SIMILAR BODIES.

Let $A B$, and $a b$, be any two similar bodies whose axes O , and o , are similarly situated. Let the linear dimensions of the larger body be n times those of the smaller. Taking similar parts at P , and p , so that P is n times as large as p in each direction, it is evident that their masses will be in the proportion of $n^3 : 1$.

i.e., *Mass of element at P : Mass of corresponding element at p*
 $= n^3 m : m$. Also, if $o p = x$, then $O P = n x$.

\therefore *Mom. of inertia of $A B$ about O* $= \Sigma n^3 m \times (n x)^2 = n^5 \Sigma m x^2$,
 and, *Mom. of inertia of $a b$ about O* $= \Sigma m x^2$.

\therefore $\left. \begin{array}{l} \text{Mom. of inertia of } A B : \\ \text{Mom. of inertia of } a b \end{array} \right\} = n^5 : 1. \dots (XVII)$

Thus, if two flywheels are made from the same drawing, but the scale in the one case be 4 inches to the foot, and in the other $1\frac{1}{2}$ inches to the foot, then their like linear dimensions will be inversely as the scales to which they are drawn, that is:—

Size of first wheel : Size of second wheel $= 1\frac{1}{2} : 4 = 3 : 8$.

\therefore *Mom. of inertia of first wheel : Mom. of inertia of second wheel* $= 3^5 : 8^5 = 243 : 32768 = 1 : 134.8$ nearly.

Centripetal and Centrifugal Force.—If a body is observed to be moving in a curvilinear path, either with uniform or variable speed, we at once infer that it is being continually acted upon by some deviating force directed towards the inside of the curve. In the case of a body moving in a circular path, that deviating force must be directed towards the centre of the circle. Hence, a body may be made to move in a circular path either by having it attached to a fixed point (the centre) by an inextensible string, or by compelling it to move in a circular groove. The necessary deviating force is supplied in the first case by the string attached to the body, while in the second case it is supplied by the sides of the groove. In either case this centrally-directed force is called the **Centripetal Force**, while its reaction is called the **Centrifugal Force**. These terms may be defined as follows:—

DEFINITION.—**Centripetal Force** is that force which a guiding body exerts on a revolving body in order to compel the revolving body to move in its curvilinear path, and is always directed towards a fixed centre.

DEFINITION.—**Centrifugal Force** is the force with which a revolving body reacts on the body that constrains it to move in a curved path, and is equal and opposite in direction to the force with which the constraining body acts on the revolving body.

i.e., **Centripetal Force = Centrifugal Force.**

We stated in Lecture X. that when the velocity of a body changes, whether in magnitude or in direction, the velocity is said to be accelerated, and we have there shown how to measure this acceleration in the case of a particle moving with uniform speed in a circle. Thus, the radial or centripetal acceleration is there shown to be:—

$$a = \frac{v^2}{r}.$$

Where, v = Linear velocity of the particle in the circle,
and, r = Radius of the circle.

But an acceleration of a body can only be produced by the action of some force on it, and in the last Lecture we have shown how this force is measured when the weight of the body and the acceleration are known. Hence:—

$$F = \frac{w}{g} a.$$

Let w = Weight of particle moving uniformly in a circle.

„ v = Linear velocity of particle in circle.

„ r = Radius of circle.

„ F = Centripetal or centrifugal force.

„ a = Centripetal acceleration = $\frac{v^2}{r}$.

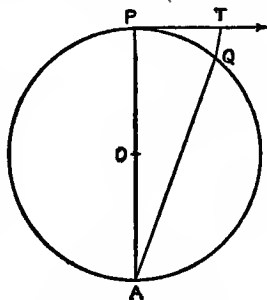
Then,
$$F = \frac{w}{g} a = \frac{w v^2}{g r} \quad \text{(XVIII)}$$

We may, however, establish the same result in a different manner as follows :—

Let P be the position of the particle at any instant, and Q its position after a small interval of time, t . If no force acted on the body during that small interval of time, it would move along the tangent PT , and at the end of the interval be found at T , such that :—

$$PT = vt.$$

But Q is its actual position ; therefore TQ represents the deviation due to the centripetal force during that interval of time. Join QA .



CENTRIPETAL FORCE.

Then,
$$TQ = \frac{1}{2} a t^2.$$

But, since PT and QT are very small, TQA will be very nearly a straight line.

$$\therefore PT^2 = TA \times TQ \quad [\text{Euc. III., 36.}]$$

$$,, = (QA + TQ) \times TQ$$

$$,, = QA \times TQ + TQ^2.$$

In the limit, when t is infinitely small and, therefore, Q infinitely near to P , we may neglect TQ^2 , and put $QA = PA = 2r$.

$$\therefore v^2 t^2 = 2r \times \frac{1}{2} a t^2,$$

$$\therefore a = \frac{v^2}{r}.$$

This is the same result as obtained by means of the Hodograph in Lecture X.

$$\therefore \text{Centrifugal force} = \frac{w}{g} a = \frac{w v^2}{g r}.$$

Let ω = Angular velocity of radius O P.

Then, $v = \omega r,$

$$\therefore F = \frac{w \omega^2 r^2}{g r} = \frac{w \omega^2 r}{g}. \quad \dots \quad (\text{XIX})$$

This shows that the centrifugal force is proportional to the square of the angular velocity of the particle, and to its distance from the centre of rotation.

We may now show that a similar expression holds good for the case of an extended rigid body turning about an axis.

Taking any particle of the body of weight w , and at a distance x from the axis of rotation, we get :—

$$\text{Cent. force of the element} = \frac{w \omega^2 x}{g}$$

$$\therefore \text{Cent. force of whole body} = \frac{\omega^2}{g} \Sigma w x.$$

$$\text{But,} \quad \Sigma w x = W r.$$

Where W = Weight of body,

And r = Distance of centre of gravity of body from axis of rotation.

$$\therefore F = \frac{W \omega^2 r}{g}. \quad \dots \quad (\text{XX})$$

Hence, if the axis of rotation passes through the centre of gravity of the body, the centrifugal force is *nil*. If, however, the body be unsymmetrical about the axis of rotation, there may be, as explained in the next Lecture, a centrifugal *couple* tending to twist the axis of rotation and make the body rotate about some other axis.

EXAMPLE VIII.—A railway carriage weighing 4 tons is moving at the rate of 60 miles per hour round a curve $\frac{1}{4}$ mile in radius. Find the pressure on the rails due to centrifugal force; also, how much the outer rail should be higher than the inner rail in order that the pressure may be equally distributed on both? The distance between the rails is 4 feet $8\frac{1}{2}$ inches.

ANSWER.—Here, $W = 4 \times 2240$ lbs.; $r = \frac{1}{4} \times 5280 = 1320$ feet; $v = \frac{60 \times 5280}{60 \times 60} = 88$ ft. per sec.

$$\therefore \left. \begin{array}{l} \text{Centrifugal} \\ \text{force} \end{array} \right\} = \frac{W v^2}{g r} = \frac{4 \times 2240 \times 88 \times 88}{32 \times 1320} = 1642.7 \text{ lbs.}$$

Hence, if both the inner and the outer rails were on a level, the flanges of the wheels would press on the latter with a force of 1642·7 lbs. By raising the outer line of rails above the level of the inner one, the carriage may be made to lie on an incline, and the outer rails thus relieved of the centrifugal pressure.

Let h = Height of the outer rail above level of the inner rail.

„ l = Distance between the rails = 4 ft. 8½ ins. = 56½ ins.

„ F = Centrifugal force on carriage = 1642·7 lbs.

Then, as a question on the *Inclined Plane*, we get:—

$$F : W = h : l.$$

$$h = \frac{F}{W} \times l = \frac{1642 \cdot 7}{4 \times 2240} \times 56\frac{1}{2} = 10 \cdot 4 \text{ inches nearly.}$$

EXAMPLE VIIIa.—An engine is running on level rails round a curve of radius, r , the distance between the rails is d , and the height of the centre of gravity above the rails is h . Prove that if the velocity exceeds $\sqrt{\frac{g r d}{2 h}}$ the engine will fall.

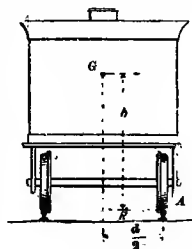
Show whether in general for a particular carriage it would be safer for the engine to push or pull a train.

ANSWER.—When a body of weight, W , or mass, M , is moving in a circle, the normal force acting at every instant to deflect it from its straight-line path (usually called the centrifugal force) is given by

$$F = \frac{W v^2}{g r}, \text{ or } \frac{1}{2} \frac{M v^2}{r}.$$

(Where W denotes the weight and M the mass of the body, v the velocity of the body in feet per second, r is the radius of the circular path in feet, and g the acceleration due to gravity.) To balance this there must be a force supplied by the flange against the rail, A ; denote this by R .

The weight, W , of the engine may be supposed to act at its centre of gravity, G . Then, taking moments about A , the moment of F is $F \times h$, and the moment of W is $W \times \frac{d}{2}$.



So long as the moment of the weight is greater than the

moment of F , equilibrium is stable. When Fh is greater than $W \times \frac{d}{2}$, the engine will overturn, and is just on the point of overturning when $Fh = W \times \frac{d}{2}$, and this obviously gives the limiting speed.

Thus we have $F \times h = W \times \frac{d}{2}$.

And substituting for F ,

$$\frac{W v^2}{g r} \times h = W \times \frac{d}{2}$$

$$\therefore v = \sqrt{\frac{g r d}{2 h}}.$$

If the engine is pushing a given carriage, the pressure on the outer rail, A , is increased, and, therefore, the tendency to overturn. The reverse occurs during the process of pulling; hence it is safer for the engine to pull than to push.

The preceding is probably the simplest method of treating the problem; more accurately, in addition to the centrifugal action, there is the gyroscopic action of the rotating wheels to be taken into account, this gyroscopic action increasing the pressure on the outer rail and diminishing the pressure on the inner, and, therefore, increasing the tendency to overturn the engine.

EXAMPLE VIIIb.—A motor car moves in a horizontal circle of 300 feet radius. The track slopes sideways at an angle of 10° with the horizontal plane. A plumb line on the car makes an angle of 12° with what would be a vertical line on the car if it were at rest on a horizontal plane; what is the speed of the car? If the car is just not side-slipping, what is the coefficient of friction? (B. of E., S. 3, 1904.)

ANSWER.—

Let v = Velocity of the car in feet per second.

„ F = Centrifugal force.

„ W = The weight of the car in lbs.

„ r = The radius of curvature of the car's motion.

„ g = The acceleration due to gravity = 32 ft. per sec. per sec.

„ μ = Coefficient of friction.

Then, the forces which are acting at the centre of gravity of the car are as follows:—

(1) The weight of the car or W lbs. acting vertically downwards.

(2) The centrifugal force F , equal to $\frac{W v^2}{g r}$ lbs. acting in a horizontal direction.

To find the inclination of the angle θ , which the resultant makes with the vertical line or direction of weight W , we have—

$$\tan \theta = \frac{\text{Magnitude of centrifugal force}}{\text{Weight of the motor car}} = \frac{F}{W}.$$

$$\text{Or,} \quad \tan \theta = \frac{Wv^2}{gr} \div W = \frac{v^2}{gr}.$$

Since W , the weight of the moving body, does not occur in this last expression v^2/gr , it follows, that the direction of the resultant force acting on the plumb will be parallel to the direction of the resultant force acting on the motor car, and that the inclination of this force to the vertical will be given by the above equation.

$$\text{Hence,} \quad v^2 = gr \tan \theta.$$

By the question, the plumb line makes an angle of $10^\circ + 12^\circ = 22^\circ$ with the vertical. Therefore, substituting the numerical values in the equation, we get—

$$v = \sqrt{gr \tan \theta} = \sqrt{32 \times 300 \times \tan 22^\circ}$$

$$\text{That is,} \quad v = \sqrt{32 \times 300 \times .375} = 60 \text{ feet per second.}$$

$$\text{Or,} \quad v = 40.9 \text{ miles per hour.}$$

The resultant force R , acting on the car at its centre of gravity makes an angle of 12° with the normal to the track. Consequently, if we resolve this resultant force into its two components; one parallel to the surface of the track, and the other normal to the surface of the track, we have from Lecture V., Vol. I.—

$$\text{Magnitude of component parallel to the surface of track} = R \sin 12^\circ.$$

$$\text{Magnitude of component normal to the surface of track} = R \cos 12^\circ.$$

$$\therefore \quad \text{Coefficient of friction } \mu = \frac{R \sin 12^\circ}{R \cos 12^\circ} = \tan 12^\circ = .208.$$

EXAMPLE VIIIc.—A train, 100 tons gross, fitted with continuous brakes, is to be run on a level line between stations one-third of a mile apart, at an average speed of 12 miles per hour, including two-thirds of a minute stop at each station. Prove that the weight on the driving wheels must exceed $22\frac{1}{2}$ tons, with an adhesion of one-sixth; neglecting road resistance and delay in application of brakes. Prove that the line could be worked principally by gravity if the road is curved downwards between the stations to a radius of about 11,740 feet, implying a dip of 33 feet between the stations, a gradient at the stations of 1 in 13, and a maximum velocity of 31 miles per hour. (London Univ. Inter. B. Sc. Eng., 1903.)

ANSWER.—(a) The statement in the question that the train is fitted with continuous brakes implies, that the brake has a shoe fitted to each wheel of the train. The tangential force on the rim of each wheel produces an equal force on the axle, retarding the train. Let this force be equal to P_2 lbs. When the brakes are on, the retarding force P_2 , will be equal to one-sixth of the weight of the train as given by the question.

Let P_1 lbs. be the direct total pull of the engine. If W_1 is the weight of the locomotive on its driving wheels, then P_1 is equal to $W_1 \div 6$.

$$\begin{aligned}
 s_1 + s_2 &= \frac{M v^2}{2 P_1 g} + \frac{M v^2}{2 P_2 g} \\
 s &= \frac{M v^2}{2 g} \left(\frac{1}{P_1} + \frac{1}{P_2} \right) = \frac{M v^2}{2 g} \left(\frac{P_2 + P_1}{P_1 P_2} \right) \\
 \left(\frac{P_1 P_2}{P_1 + P_2} \right) \cdot s &= \frac{M v^2}{2 g} \quad \dots \quad (7)
 \end{aligned}$$

Dividing equation (7) by equation (4), we get—

$$\frac{\left(\frac{P_1 P_2}{P_1 + P_2} \right) \cdot s}{\left(\frac{P_1 P_2}{P_1 + P_2} \right) \cdot t} = \frac{\frac{M v^2}{2 g}}{\frac{M v}{g}} \quad \therefore \quad \frac{s}{t} = \frac{v}{2} \quad \dots \quad (8)$$

This result might have been obtained from the consideration that during the first t_1 seconds, the velocity of the train increases uniformly from 0 to v so that during this interval the average velocity is equal to $v/2$. During the second interval t_2 , the velocity decreases uniformly from v to 0, so that in this interval the average velocity is also equal to $v/2$. Therefore, the average velocity from start to stop—i.e., s/t —must be equal to $v/2$.

The booked time between the stations ($\frac{1}{2}$ mile apart) when the average speed is 12 miles per hour is 100 seconds. Deducting 40 seconds for the stop at the station, then the actual running time is 60 seconds.

Substituting numerical values in equation (8), we get—

$$v = \frac{2s}{t} = \frac{2 \times 1,760}{60} = \frac{176}{3} \text{ feet per second.}$$

From equation (4)—

$$\begin{aligned}
 t &= \frac{M v}{g} \left(\frac{1}{P_1} + \frac{1}{P_2} \right) \\
 \therefore \quad \frac{g t}{v} &= M \left(\frac{1}{P_1} + \frac{1}{P_2} \right) = \frac{M}{P_1} + \frac{M}{P_2} \quad \dots \quad (9)
 \end{aligned}$$

Since each of the quantities on the right of this last equation is a ratio, we can easily measure M_1 , P_1 , and P_2 in tons, instead of in pounds, without incurring any error.

Then, in equation (9), $M = 100$ tons, $P_2 = \frac{100}{6}$ tons, $g = 32$ feet per second per second, $t = 60$ seconds, and $v = \frac{176}{3}$ feet per second.

Substituting these values, we get—

$$\begin{aligned}
 \frac{M}{P_1} &= \frac{g t}{v} - \frac{M}{P_2} = \left(\frac{32 \times 60}{\frac{176}{3}} - \frac{100}{\frac{100}{6}} \right) \\
 \text{Or,} \quad \frac{M}{P_1} &= \frac{32 \times 60 \times 3}{176} - 6 = \frac{5,760 - 1,056}{176} = \frac{4,704}{176} \\
 \therefore \quad P_1 &= \frac{176 \times M_1}{4,704} = \frac{176 \times 100}{4,704} = \frac{1,100}{294} \text{ tons.}
 \end{aligned}$$

The total weight W_1 on all the driving wheels must be equal to $6 P_1$

$$\therefore \quad W_1 = 6 P_1 = \frac{6 \times 1,100}{294} = \frac{1,100}{49}$$

Or, $W_1 = 22.45$; or, 22.5 tons (approximately).

ANSWER.—(b) Suppose the line to be worked by gravity.

Let l = the constant radius of curvature of the road in a vertical plane, when the starting and stopping stations are on a level.

Then, neglecting friction, the time t , required for a train to travel from one station to the other will be equal to the time required for a pendulum, of length l , to complete one-half of a complete to and fro swing or half one oscillation. Referring to any book on the properties of the pendulum, we get—

$$t = \pi \sqrt{\frac{l}{g}}; \quad \therefore l = \frac{t^2 g}{\pi^2}. \quad (10)$$

Now, since $t = 60$ seconds, and $g = 32.2$ feet per second per second, we obtain from equation (10)—

$$l = \frac{t^2 g}{\pi^2} = \frac{60 \times 60 \times 32.2 \times 7 \times 7}{22 \times 22}$$

$$l = 11,740 \text{ feet.}$$

Let h be the dip and s the distance between the stations.

Then, by Euclid and the accompanying figure—

$$AO^2 = AD^2 + DO^2,$$

$$\text{i.e.,} \quad l^2 = \left(\frac{s}{2}\right)^2 + (l - h)^2.$$

$$\therefore 2lh - h^2 = \left(\frac{s}{2}\right)^2. \quad (11)$$

Neglecting h^2 in comparison with $2lh$, we obtain from equation (11)—

$$h = \frac{s^2}{4 \times 2l} = \frac{(1,760)^2}{8 \times 11,740} = 33 \text{ feet.}$$

At the stations, the road is inclined downwards at an angle, of which the circular measure is practically equal to—

$$\frac{s}{2} / l = \frac{s}{2l} = \frac{880}{11,740} = \frac{1}{13.3}$$

Thus, the gradient is practically 1 in 13.

In a simple harmonic motion, the maximum velocity is obtained by multiplying the amplitude by $2\pi/T$, where T is the time of a complete oscillation. The amplitude in the present case is $s/2 = 880$ feet, while $T = 2 \times 60$ seconds.

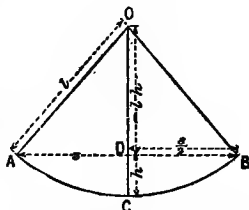
$$\therefore \text{Maximum velocity } V = \frac{2\pi}{T} \cdot \frac{s}{2} = \frac{2\pi}{2 \times 60} \times 880 = \frac{22 \times 88}{7 \times 6} = 46 \text{ ft. per sec.}$$

The maximum velocity V may also be determined from the formula—

$$V^2 = 2gh = 2 \times 32 \times 33 = 64 \times 33.$$

$$\therefore V = 8\sqrt{33} = 8 \times 5.744 = 45.95 \text{ feet per second.}$$

$$\text{Or,} \quad V = 46 \times 60 \times 60 \div 5,280 = 31.36 \text{ miles per hour.}$$



APPLICATION OF THE PENDULUM TO FIND h IN TERMS OF s AND l .

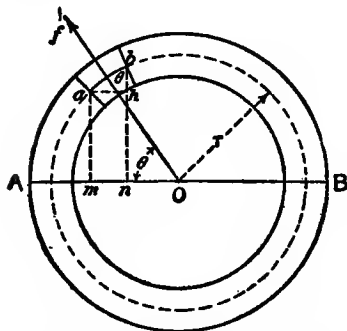
Straining Actions due to Centrifugal Forces.—Whenever a body rotates about an axis, the material of that body becomes strained by reason of the centrifugal force set up. Thus, in the case of a flywheel or pulley, the centrifugal forces set up may be sufficient to tear the rim from the arms, the arms from the nave, or to burst the rim. The effects of the centrifugal forces acting on a belt when moving over a pulley with a high velocity are explained in Vol. V., "Theory of Machines." We there show that the tensions in the two parts of the belt are increased by the centrifugal action on that part of the belt which is in contact with the pulley. We shall now show that similar effects occur in a rapidly-revolving flywheel or pulley.

Suppose we have a flywheel built up of segments, each segment being attached to an arm, while they are also attached to each other by dowels and cotters, or bolts, &c. Let the weight of each segment be W ; the distance of its centre of gravity from the axis of rotation, r , and the angular velocity of the wheel, ω . Then, neglecting the assistance afforded by the connection between the various segments, it is obvious that the tension in the arm to which the segment is attached is :—

$$P = \frac{W \omega^2 r}{g},$$

The arm must, therefore, be made strong enough to withstand this stress.

Again, in the case of a solid rim, the effect of the centrifugal forces is to burst it along a section made by a plane containing the axis of the shaft. Let the figure represent the rim of a flywheel. Then, in order to calculate the stress in its material at any section, AB , made by a plane containing the axis, O , consider the effects of a thin slice of the rim at $a b$.



Let W = Total weight of rim.

„ r = Mean radius of rim.

„ x = Length of small arc ab of mean rim.

„ ω = Angular velocity of wheel.

STRESS IN RIM OF FLYWHEEL DUE TO CENTRIFUGAL FORCE.

$$\text{Then, } \frac{\text{Weight of slice } a b}{\text{Weight of rim}} = \frac{\text{Arc } a b \text{ of mean rim}}{\text{Circumference of mean rim}} = \frac{x}{2 \pi r}.$$

$$\therefore \text{Weight of element } a b = \frac{W}{2 \pi r} \times x.$$

The centrifugal force of the element at $a b$ is:—

$$\therefore f = \frac{W}{2 \pi r} \times \frac{\omega^2 r}{g} \times x = \frac{W \omega^2}{2 \pi g} x.$$

This force acts through the *c.g.* of the element. Resolve f in directions parallel and perpendicular to $A B$. The latter component only is effective in producing stress at the sections A and B .

$$\therefore \left. \begin{array}{l} \text{Stress at sections } A \text{ and } B \text{ due} \\ \text{to cent. force on element } a b \end{array} \right\} = f \sin \theta = \frac{W \omega^2}{2 \pi g} \times x \sin \theta.$$

Where, $\theta = \angle A O f$.

From a and b drop the perpendiculars $a m$, $b n$ on $A B$, and through a draw $a h$ perpendicular to $b n$. Then $\angle a b h = \theta$ and $x \sin \theta = a h = m n$.

$$\therefore \left. \begin{array}{l} \text{Stress at sections } A \text{ and } B \text{ due} \\ \text{to cent. force on element } a b \end{array} \right\} = \frac{W \omega^2}{2 \pi g} \times m n.$$

Continuing this reasoning for all the slices from A round to B , and adding the results, we get:—

$$\left. \begin{array}{l} \text{Total stress over} \\ \text{sections at } A \\ \text{and } B \end{array} \right\} = \frac{W \omega^2}{2 \pi g} \sum m n = \frac{W \omega^2}{2 \pi g} \times 2 r = \frac{W \omega^2 r}{\pi g}.$$

Let A = Area of section of rim at A , or B , in *square inches*.

„ p = Stress in *lbs. per square inch* over section.

„ w = Weight of a *cubic foot* of material of rim.

$$\text{Then, } p = \frac{\text{Total stress over section at } A \text{ or } B}{\text{Area of section}}.$$

$$\text{Or, } p = \frac{\frac{1}{2} \times \frac{W \omega^2 r}{\pi g}}{A} = \frac{W \omega^2 r}{2 \pi g A}.$$

But, W = Area of cross section of rim in *sq. ft.* $\times 2 \pi r w$

$$\therefore W = \frac{A}{144} \times 2 \pi r w.$$

Substituting this in the last equation, we get :—

$$p = \frac{w \omega^2 r^2}{144 g} \dots \dots \dots \text{(XXI)}$$

Or, if n = Revolutions of wheel *per second*,

d = Diameter of rim *in feet*,

v = Velocity of rim *in feet per second* = ωr .

$$\left. \begin{array}{l} \text{Then,} \quad p = \frac{w \pi^2 d^2 n^2}{144 g} \text{ lbs. per square inch.} \\ \text{Or,} \quad p = \frac{w v^2}{144 g} \quad " \quad " \quad " \end{array} \right\} \text{(XXII)}$$

From this we see that the stress per square inch does not depend on the cross area of the rim nor the diameter of the wheel, but only on the density of the material and its speed. It will also be observed that the centrifugal force in the rim is similar in effect to a hydrostatic pressure on the inside of a cylindrical vessel.

EXAMPLE IX.—A flywheel, 21 feet in diameter, makes 100 revolutions per minute. The weight of a cubic foot of its material is 448 lbs. Find the intensity of stress on a transverse section of rim, assuming that it is unaffected by the arms. If the safe stress permissible in the material is 6,000 lbs. per square inch, what is the greatest speed at which the wheel can be run with safety?

ANSWER.—Here, w = 448 lbs. per cubic foot; d = 21 feet:
 $n = \frac{100}{60} = \frac{5}{3}$ revolutions per second.

Therefore, from equation (XXII), we get :—

$$\text{Stress in rim} = p = \frac{w \pi^2 d^2 n^2}{144 g}$$

$$\text{Or,} \quad p = \frac{448 \times \left(\frac{22}{7}\right)^2 \times 21^2 \times \left(\frac{5}{3}\right)^2}{144 \times 32} = 1176.4 \text{ lbs. per sq. in.}$$

Next, let n = Maximum number of revolutions per second which the wheel can make without bursting.

$$\text{Then, from the previous formula :—} p = \frac{w \pi^2 d^2 n^2}{144 g},$$

$$\text{We get,} \quad n^2 = \frac{144 g p}{w \pi^2 d^2}, \quad \text{or,} \quad n = \frac{12}{\pi d} \sqrt{\frac{g p}{w}}.$$

Substituting $p = 6,000$, and the values for the other letters, we get:—

$$n = \frac{12}{\frac{22}{7} \times 21} \sqrt{\frac{32 \times 6,000}{448}} = 3.76 \text{ revs. per sec.} = 225.6 \text{ r.p.m.}$$

Note.—Students should refer to the author's *Text-Book on Steam and Steam Engines*, Lecture XVIII., for a discussion of the effects of the inertia of the moving parts of an engine; or, to Vol. V., "Theory of Machines" section, of this text-book.

Students who are desirous of studying special methods of arriving at the moments of inertia of different sections of bodies about different axes, should refer to Dr. R. F. Muirhead's paper on "Equipomental Systems and their Use in Applied Mechanics," and the discussion as found in the *Trans. of The Inst. of Engs. and Shipbuilders*, vol. xlix., Session 1905-6.

LECTURE XII.—QUESTIONS.

1. Define the terms moment of inertia and radius of gyration of a body. Find the moment of inertia of rectangular lamina—first, with respect to one edge; secondly, with respect to a diagonal.

2. An axis is drawn through the centre of gravity of a body whose mass is M ; a second axis is drawn parallel to the former and at a distance, h , from it. If I denotes the moment of inertia of the body with respect to the first axis, show that the moment of inertia with respect to the second axis is $I + Mh^2$. A fine wire of uniform thickness is bent into the form of a circle whose radius is r ; find its moment of inertia with respect to an axis passing at right angles to the plane of the circle through a point in the circumference. *Ans.* $\frac{3}{2}Mr^2$.

3. State and prove the theorem of moments of inertia for parallel axes. Find the moment of inertia of a cylinder about a line perpendicular to its axis through its mid point. (S. & A. Theor. Mechs. Hons. Exam.)

4. A wheel and axle are composed of the same specific gravity. The wheel is 4 feet radius, and 6 inches thick. The axle is 6 inches radius and 4 feet long. Find radius of gyration of the whole about the axis. *Ans* $k = \sqrt{7 \cdot 125} = 2 \cdot 67$ ft.

5. The rim of a flywheel is rectangular in section, 6 inches wide, outside and inside radii 6 and 5 feet respectively. The nave is cylindrical, 2 feet long and 1 foot in diameter. There are eight cylindrical spokes of 4 inches diameter. Find the radius of gyration of the wheel. *Ans.* 4·8 ft.

6. Show that the kinetic energy of a body revolving with an angular velocity, ω , about a given axis is $\frac{1}{2} I \omega^2$, where I denotes the moment of inertia of the body with reference to the axis. A flywheel has a mass of 30 tons, which may be supposed to be distributed along the circumference of a circle 8 feet in radius; it makes 20 revolutions a minute; find its kinetic energy in foot-pounds. *Ans.* 295,000 ft.-lbs.

7. Find the moment of inertia of a rectangular lamina about an edge. A rectangular lamina, whose shorter edges are 4 feet long, turns round one of its longer edges 50 times a minute. It weighs 441 lbs.; find its kinetic energy. *Ans.* 1008·3 ft.-lbs.

8. When a rigid body turns round an axis, what relation exists between its angular velocity and its kinetic energy? A rod of uniform density can turn freely round one end; it is let fall from a horizontal position; what is its angular velocity when it reaches its lowest position? Prove your equations. *Ans.* $\omega = \sqrt{\frac{3g}{l}}$.

9. How do you estimate the total energy possessed by a body when moving with both translation and rotation? Find the velocity of the centre (1) when a hoop, (2) when a disc, and (3) when a sphere rolls down

an inclined plane of height, h . *Ans.* (1) $v = \sqrt{gh}$, (2) $v = 2\sqrt{\frac{gh}{3}}$, (3) $v = \sqrt{\frac{10gh}{7}}$.

10. Sketch, and explain the principle of the action of, a fly press for stamping metals. If a velocity of 5 feet per second is given to the balls of such a press, and their motion is stopped after the screw has made one-quarter of a turn from the time that the die touches the metal, the pitch of the screw being $\frac{1}{4}$ inch; find the weight of the balls, so that the pressure exerted may be 4,000 lbs. *Ans.* 26·67 lbs. each.

11. Two weights of 100 lbs. each are placed at the ends of the arms of a fly-press, and are moving with a velocity of 12 feet per second. How many foot-pounds of work must be expended in bringing them to rest? Hence explain the mechanical action of the fly-press as a machine for punching or stamping metals. *Ans.* 450 ft.-lbs.

12. In a fly-press there are two weights, each of 60 lbs., placed at the ends of an arm which drives the screw; and the velocity of each weight at the instant of striking the blow is 10 feet per second. The die at the end of the screw moves through $\frac{1}{4}$ inch in coming to rest; what mean statical pressure does it exert on the metal subjected to the operation of stamping? *Ans.* 22,500 lbs.

13. In a fly-press for stamping metals a ball of 70 lbs. is placed at each end of the lever attached to the head of the screw. At the moment of striking the blow the weights have a velocity of 550 feet per minute, and the die at the end of the screw indents the metal to a depth of $\frac{1}{4}$ inch before coming to rest. What would be the mean statical pressure exerted on the metal? *Ans.* 26,468·75 lbs.

14. Proves that the kinetic energy of a train of railway carriages moving with velocity, v , is $\left\{ W + w \left(1 + \frac{k^2}{r^2} \right) \right\} \frac{v^2}{2g}$ ft. lbs., where w denotes the weight of the wheels and axles; W the weight of the rest of the train; r the radius of the wheels, and k the radius of gyration of a pair of wheels about their axis, the units being feet, lbs., and seconds. Determine the acceleration with which the train would freely descend an incline of inclination, α .

15. Describe and show by the necessary sketches the construction of a fly-press for punching holes in iron plates. In such a press the two balls weigh 30 lbs. each, and are placed at a radius of 30 inches from the axis of the screw, the screw itself being of 1 inch pitch. What diameter of hole could be punched by such a press in a wrought-iron plate of $\frac{1}{4}$ inch in thickness; the shearing strength of the metal being 22·5 tons per square inch? (Consider that the balls are revolving at the rate of 60 revolutions per minute when the punch comes into contact with the metal, and that the resistance of the plate is overcome in the first sixteenth of an inch of the thickness of the plate.) *Ans.* 1·12 ins.

16. A pendulum bob weighing 20 lbs. is suspended by a wire, the length from the point of suspension to the centre of the bob being 16 feet. The pendulum swings through an angle of 30° on each side of the vertical; find its potential energy when in the highest position, and its velocity when passing the lowest point. *Ans.* 42·88 ft.-lbs.; 11·71 ft. per second.

17. A flywheel weighs 10,000 lbs., and is of such a size that the matter composing it may be treated as if concentrated on the circumference of a circle 12 feet in radius; what is its kinetic energy when moving at the rate of 15 revolutions a minute? How many turns would it make before coming to rest if the steam were cut off and it moved against a friction of 400 lbs. exerted on the circumference of an axle 1 foot in diameter? *Ans.* 55,520 ft.-lbs.; 44·2 turns.

18. The sectional area of the rim of a cast-iron flywheel is 12 square

inches and the mean radius (or radius of gyration) is 25 inches; what is the kinetic energy at 150 revolutions per minute? What moment of constant magnitude, and acting through one-quarter revolution, would increase the speed to 155 revolutions per minute at the end of the quarter revolution? What would be the length of a solid wrought-iron shaft, 5 inches in diameter, rotating at the same speed and having the same kinetic energy? *Ans.* 35,707,000 ft.-lbs.

19. Prove the formula for the energy stored up in a flywheel on the supposition that the whole of the material is collected in a heavy rim of given mean radius. Apply the formula to show (1) the effect of doubling the number of revolutions per minute; (2) the effect of doubling the weight; (3) the effect of increasing the mean radius in the proportion of 3 to 2.

20. The rim of a flywheel weighs 9 tons, and the mean linear velocity of its mass is assumed to be 40 feet per second; how many foot-tons of work are stored up in it? If it be required to store the additional work of 9 foot-tons, what should be the increase of velocity? *Ans.* 225 ft.-tons; 0.79 ft. per second.

21. A flywheel weighs $2\frac{1}{2}$ tons, and its mean rim has a velocity of 40 feet per second. If the wheel gives out 10,000 foot-pounds of energy, how much is its velocity diminished? *Ans.* 1.455 ft. per second.

22. A flywheel weighing 5 tons has a mean radius of gyration of 10 feet. The wheel is carried on a shaft of 12 inches diameter and is running at 65 revolutions per minute; how many revolutions will the wheel make before stopping if the coefficient of friction of the shaft in its bearing is 0.065? (Other resistances may be neglected.) *Ans.* 354.66 turns.

23. A particle of given mass moves with a given velocity in a circle of given radius; state what is known as to the force which acts on the particle. Prove the statement. *Ans.*

24. If a locomotive weighing 55 tons runs round a curve of 1,200 feet radius at 20 miles per hour, what is its centrifugal force? How much higher in level should the outer rail be laid than the inner rail in order that the resolved part of the weight of the locomotive should balance this centrifugal force without pressure being exerted by the outer rail, the gauge being 4 feet $8\frac{1}{2}$ inches? *Ans.* 2760.6 lbs.; 1.27 inches.

25. Prove that a railway carriage running round a curve of radius, r , will upset if the velocity is greater than $\sqrt{\frac{g r a}{2 h}}$, where a is the distance between the rails, and h the height of the centre of gravity of the carriage above the rails.

26. Show that by raising the outside rail of a railway track in going round a curve the tendency of the train to leave the rails is diminished, and that if θ be the inclination of the floor of the carriage to the horizontal, when there is no lateral pressure, $\tan \theta = \frac{v^2}{g r}$, where r is the radius of the curve, and v the velocity of the train. Hence show that on a 5-foot track, round a curve of one-eighth of a mile radius, that for a mean velocity of 30 miles an hour the outside rail ought to be raised $5\frac{1}{2}$ inches above the level of the inside rail.

27. A body moves in a circle with a uniform velocity, show that it must be acted on by a constant force tending towards the centre, and find the magnitude of the force in terms of the radius of the circle, and of the

mass and velocity of the body. A body weighing $2\frac{1}{2}$ lbs. fastened to one end of a thread 4 feet long is swung round in a circle of which the thread is the radius; what will be its velocity when the tension of the thread is a force of 20 lbs.? ($g = 32$). *Ans.* 32 ft. per second.

28. A segment of a flywheel with the arm to which it is attached weighs 3,500 lbs., and the mass of the portion may be taken as collected at a distance of 8 feet from the axis of the wheel, which makes 40 revolutions per minute. What is the force tending to pull away the segment and arm from the boss of the wheel? You are required to write out a proof of the formula which you employ. *Ans.* 15,365 lbs.

29. Show that the stress per square inch on the rim of a flywheel is equal to the momentum of the amount of rim (per square inch of section) which passes a fixed point in the unit of time. Find the limiting speed of periphery, the material being such that a bar of uniform section 900 feet long may be supported by tension. *Ans.* $30\sqrt{g}$.

30. A flywheel 20 feet in diameter makes 80 revolutions per minute. Find the stress in its rim due to centrifugal forces, assuming that it is unaffected by the connection with the arms. The weight of a cubic foot of the material forming the rim is 500 lbs. What is the maximum speed at which the wheel can be safely run if the tensile strength of the material has not to exceed 6,000 lbs. per square inch? *Ans.* 762 lbs. per sq. in.; 224.5 revs. per min.

31. When the fly-wheel of a certain traction engine lessens in speed from 150 to 140 revolutions per minute, there is a loss of kinetic energy (on the motion of the whole engine as well as the fly-wheel) of 25,000 foot-pounds.

If the speed is 160 revolutions per minute, how far will the engine travel up an ascent of 1 in 100 before coming to rest, if engine and truck together weigh 30 tons, and there is a constant frictional resistance on a level road of 20 lbs. to the ton? *Ans.* 173.5 feet.

32. The centre of gravity of a body of 100 lbs. is revolving at 15 inches from an axis, at 250 revolutions per minute. What is its centrifugal force? Prove the Rule. *Ans.* 2,663 lbs.

33. Describe experiments to compare the speed of a flywheel with the work given to it by a falling weight. If you have not made such experiments, so that the exact method of finding the speed, of correcting for friction, &c., are unknown to you, you had better not attempt this question.

34. A flywheel is required to store 12,000 ft.-lbs. of energy as its speed increases from 98 to 102 revolutions per minute; what is its moment of inertia? The wheel is a solid disc of cast iron, its thickness one-tenth of its diameter, what is its diameter? Prove the formula you use for calculating its moment of inertia. *Ans.* 86,400 ft.-lbs.; 14.3 feet.

(B. of E. H., Part I., 1900.)

35. A flywheel of a shearing machine has 150,000 foot-pounds of kinetic energy stored in it when its speed is 250 revolutions per minute, what energy does it part with during a reduction of speed to 200 revolutions per minute? If 82 per cent. of this energy given out is imparted to the shears during a stroke of 2 inches, what is the average force due to this on the blade of the shears? (B. of E. Adv., 1902.)

36. A flywheel weighs 5 tons and its radius of gyration is 6.30 feet, what is its moment of inertia in engineers' units? It is at the end of a shaft 40 feet long, 5 inches diameter, modulus of rigidity of material 12×10^6 lb.-inches, what is the natural time of torsional vibration of the system, neglecting the inertia of the shaft itself? (B. of E. H., Part I., 1902.) *Ans.* $I = 13,980$ units; time of torsional vibration = 0.3 second.

N.B.—See Appendices B and C for other questions and answers.

LECTURE XII.—A.M.INST.C.E. EXAM. QUESTIONS.

1. Explain the importance of elevating the exterior rail in railway curves. Calculate the proper slope for a circular bicycle track 100 feet in radius, to prevent any tendency to side-slip at a speed of 24 miles an hour. (I.C.E., Oct., 1897.)

2. In a gas-engine using the Otto cycle the I.H.P. is 8 and the speed is 264 revolutions per minute. Treating each fourth single stroke as effective and the resistance as uniform, find how many foot-pounds of energy must be stored in the flywheel in order that the speed shall not vary by more than one-fortieth of its mean value. (I.C.E., Oct., 1897.)

3. A flywheel supported on a horizontal axle 2 inches in diameter is pulled round by a cord on the axle, carrying a weight. It is found that a weight of 4 lbs. is just sufficient to overcome the friction. A further weight of 16 lbs., making 20 in all, is applied, and after 2 seconds (starting from rest) it is found that the weight has gone down 12 feet. Find the moment of inertia of the wheel. (I.C.E., Oct., 1897.)

4. A brake wheel 4 feet in diameter on a horizontal axle is furnished with internal flanges, which, along with the rim, form a trough containing cooling water. What is the least speed that will prevent the water from falling? (I.C.E., Feb., 1898.)

5. A flywheel alters in speed from 99 to 101 revolutions per minute, when its kinetic energy alters by the amount of 500,000 foot-lbs.; what is its moment of inertia? What is its kinetic energy when making 1 revolution per minute? (I.C.E., Oct., 1898.)

6. Find the radius of gyration in a hollow cylindrical column with an external diameter of 12 inches and a thickness of 1 inch. Also in a solid square column 4 inches by 4 inches. (I.C.E., Oct., 1898.)

7. A small flywheel is mounted with its axle vertical. Over a small pin on the axle is hooked the looped end of a stout cord, and the wheel is turned round until there are ten coils of the cord on the axle. The cord is led over a small pulley and a weight of 20 lbs. attached to its end. This weight is now allowed to fall, giving motion to the flywheel. It is found that after the lapse of 8 seconds the cord slips off the pin, the weight having fallen 4 feet during this time. During the fall the force of gravity has done work; in what form does this appear at the instant the cord leaves the axle of the wheel? Suppose that 85 per cent. of this energy is stored in the flywheel, calculate its moment of inertia. Describe the nature of the experiments you would make to measure any loss of mechanical energy which occurs. (I.C.E., Feb., 1899.)

8. Find the moment of inertia of a circular disc about a tangent, and of a square disc about an axis through one corner perpendicular to its plane. (I.C.E., Oct., 1899.)

9. Define moment of inertia. Find the moment of inertia of an equilateral triangle about its base and about an axis through its apex perpendicular to its plane. (I.C.E., Feb., 1900.)

10. A cast-iron flywheel 10 feet in diameter, with a rim 6 inches by 6 inches, is rotating freely on a shaft at the rate of 100 revolutions per minute. A brake, which exerts a frictional retardation of 100 lbs., is applied to its rim for 20 seconds. Find how much the speed of the flywheel is reduced (neglecting the weight of its arms). (I.C.E., Feb., 1900.)

11. Define "Moment of inertia," and prove the formula which expresses the kinetic energy stored in a body rotating about a fixed axis. How much energy is stored in a 3-foot thin rod weighing 4 lbs., and which is re-

volving at 140 revolutions per minute about an axis through its centre and perpendicular to its length? (I.C.E., Oct., 1900.)

12. The flywheel of an engine of 4 H.P. running at 75 revolutions per minute is equivalent to a heavy rim 2 feet 9 inches mean diameter and weighing 500 lbs. Determine the maximum and minimum speeds of rotation when the fluctuation of energy is one-fourth the energy of a revolution. (I.C.E., Oct., 1900.)

13. A flywheel weighing 5 tons has a radius of gyration of 10 feet. The wheel is carried on a shaft of 12 inches in diameter, and is running at sixty-five revolutions per minute. How many revolutions will the wheel make before stopping if the coefficient of friction of the shaft in its bearings is 0.065? (I.C.E., Oct., 1901.)

14. A small cast-iron flywheel weighs 96 lbs. It is mounted between conical bearings, and so arranged that a falling weight causes it to rotate, the weight being at the end of a cord, which is wound round the spindle of the wheel. The falling weight is 21 lbs., and acts upon the wheel during a fall of $3\frac{1}{2}$ feet, its velocity at the end of the fall being 0.595 foot per second. If the work done in overcoming the friction of the wheel spindle during the fall of the weight is 12.726 foot-lbs., find the kinetic energy of the flywheel at the instant the falling weight ceases to act upon it.

(I.C.E., Feb., 1902.)

15. Define "centrifugal force." A locomotive engine weighs 38 tons, and travels round a curve of 800 feet radius at a speed of 50 miles an hour. Find the centrifugal force. Show also how to find: (a) the direction and magnitude of the resultant thrust on the rails due to the weight of the engine and the centrifugal force; (b) the height to which the outer rail should be raised over the inner rail, in order that the plane of the rails may be perpendicular to this resultant. Gauge of rails 4 feet 8 $\frac{1}{2}$ inches.

(I.C.E., Feb., 1902.)

16. Explain the use of a flywheel. Determine the weight of rim per horse-power which, when running at a speed of 70 feet per second, will have stored in it 10 per cent. of the energy developed per minute.

(I.C.E., Oct., 1902.)

17. Explain the meaning of the term "centrifugal force." With what speed must a locomotive be running on level railway lines, forming a curve of 968 feet radius, if it produce a horizontal thrust on the outer rail equal to $\frac{1}{4}$ of its weight? (I.C.E., Feb., 1903.)

18. A factory is driven by an engine working at 100 I.H.P. One machine requiring 3 H.P. is thrown out of gear, and in two minutes the speed of the engine has increased from 40 to 43 revolutions per minute. The engine power remaining constant and the surplus work being accumulated in the flywheel rim, find the weight of the rim, its diameter being 15 feet. (I.C.E., Feb., 1903.)

N.B.—See Appendices B and C for other questions and answers.

APPENDICES.

APPENDIX A (p. 334 to p. 343).

- (i.) General Instructions by The Board of Education for their Examinations on Applied Mechanics, with Syllabus for Stages 2 and 3.
- (ii.) General Instructions by The City and Guilds of London Institute for their Examination on Mechanical Engineering, with Syllabus.
- (iii.) Rules and Syllabus of Examinations by The Institution of Civil Engineers for Election of Associate Members.

APPENDIX B (p. 344 to p. 367).

- (i.) Board of Education's Exam. Papers in Applied Mechanics, Stages 2 and 3. The City and Guilds of London Institute's Honours Examinations in Mechanical Engineering. And, The Institution of Civil Engineers' Examination Papers in Applied Mechanics, arranged in the order of the Lectures.
- (ii.) All new Answers to Questions for the respective Lectures are tabulated under the two main headings—*either* Board of Education and City and Guilds *or* Institution of Civil Engineers.

APPENDIX C (p. 368 to p. 382).

The latest Exam. Papers pertaining to Applied Mechanics and set by the governing bodies enumerated under Appendix A.

APPENDIX D (p. 383 to p. 392).

The Centimetre, Gramme, Second, or C.G.S. System of Units of Measurement and their Definitions—Fundamental Units—Derived Mechanical Units—Practical Electrical Units.

Tables of Constants, Logarithms, Antilogarithms, and Functions of Angles.

APPENDIX A.

MAY EXAMINATION ON SUBJECT VIIA. APPLIED MECHANICS.*

BY THE BOARD OF EDUCATION SECONDARY BRANCH,
SOUTH KENSINGTON, LONDON.

GENERAL INSTRUCTIONS.

If the rules are not attended to, your paper will be cancelled.

*Immediately before the Examination commences, the following
REGULATIONS are TO BE READ TO THE
CANDIDATES.*

Before commencing your work, you are required to fill up the numbered slip which is attached to the blank examination paper.

You may not have with you any books, notes, or scribbling paper.

You are not allowed to write or make any marks upon your paper of questions, or to take it away before the close of the examination.

You must not, under any circumstances whatever, speak to or communicate with one another, and no explanation of the subject of examination may be asked or given.

You must remain seated until your papers have been collected, and then quietly leave the examination room. None of you will be permitted to leave before the expiration of one hour from the commencement of the examination, and no one can be re-admitted after having once left the room.

Your papers, unless previously given up, will all be collected at 10 o'clock.

If any of you break any of these rules, or use any unfair means, you will be expelled, and your paper cancelled

**Before commencing your work, you must carefully
read the following instructions:—**

Candidates who have applied for examination in the Elementary Stage must confine themselves to that stage. Candidates who have not applied to take the Elementary Stage may take Stage 2, or Stage 3, or, if eligible, Honours, but they must confine themselves to one of them.

Put the number of the question before your answer.

You are to confine your answers *strictly* to the questions proposed.

Such details of your calculations should be given as will show the methods employed in obtaining arithmetical results.

A table of logarithms and functions of angles and useful constants and formulæ is supplied to each candidate. (See end of Appendix to this Book.)

The examination in this subject lasts for three hours.

* See Appendix C for the latest Exam. Papers.

Notes re Board of Education Examinations on Applied Mechanics.

This subject has been divided. The first division (a) is arranged so as to meet the special needs of students engaged in the Building Trades, in the offices of Civil Engineers, or on cognate work. The subjects dealt with in this division will mainly consist of those branches of Applied Mechanics usually denoted by the terms *Strength of Materials and Structures*.

The second division (b) deals with those branches of Applied Mechanics included under the terms *Theory of Machines and Hydraulics*. Mechanical Engineering students will study the subjects of both divisions.

In these divisions there must of necessity be a little overlapping of the work; for example, it is essential that building students should have some knowledge of the principles of such subjects as work, friction, the efficiency of lifting machinery, and of similar branches of the subject.

There will be only one examination paper in Stage 1, but the questions will be arranged in two series, corresponding to the divisions (a) and (b). Considerable latitude will be allowed to students taking the examination in Stage 1 of this subject in selecting their questions from these two series. A candidate will be allowed to select a majority of questions from one of the two series, but he will be expected to answer some questions also from the other. In Stage 2 and in Stage 3 there will be two examinations on separate evenings—one in division (a) and another in division (b). In Honours there will be only one examination paper.

It is understood that in all the stages, and particularly in the higher stages, candidates must exhibit some knowledge of the difficulty of applying academic theory to practical problems. Compulsory questions may be set in any of the examination papers.

Subject vii. (a).—Strength of Materials and Structures.

STAGE 2.

Candidates are expected to have an advanced knowledge of the subjects of Stage 1. Thus, in *Graphical Statics*, they must know how to find the resultant of forces which do not meet at a point. The forces in hinged structures (see the Author's Vol. III. on *Graphic Statics and Theory of Structures*). A better knowledge of *materials* when tested to destruction in the laboratory; sudden loading; initial stresses and strains, copper and its alloys. Experimental measurement of moduli of elasticity. They must really know what is meant by *bending moment* at a section. It is quite usual to find a candidate, who, when given the loads, can draw diagrams of bending moment and shearing force, yet when he is asked "What do you mean by bending moment?" is unable to answer. They must be able to draw such diagrams, and to calculate the tensile and compressive *stresses* at all places in a section, and to determine centres of area, moments of inertia and resistance. An elementary knowledge of the strength of *re-inforced concrete* will be expected. In *struts* and pillars, an elementary knowledge of the effect of the form of the cross-section and the length of the strut or pillar, and the effect of fixing the ends. Determination of stiffness of spiral and carriage springs by experiment. The design of simple cottered *joints*, and of riveted joints in single and double shear. Lines of resistance in *arches* may be treated as inverted chains, after which

the student may be given an elementary knowledge of what occurs in masonry and metal arches. They will be expected to apply their mathematics to easy examples on work, power, velocity, acceleration, and force (see the Author's Vol. II. on *Strength and Elasticity of Materials*).

STAGE 3.

Candidates are expected to have a more advanced knowledge of the subjects of Stages 1 and 2.

Graphical Statics.—Candidates are expected to be able to use their descriptive geometry to find the resultant of forces not in one plane, or the forces in parts of a structure not in one plane. Force diagrams to determine the forces in the various members of different types of roof trusses and built up girders due to dead and to live loads. The principal moments of inertia of an area (see the Author's Vol. III. on *Theory of Structures*).

Testing of Materials.—Testing of metals in tension, compression, and shear. Influence of the shape of the test-piece in the testing of timber, iron, and steel. The testing of cement for tenacity and compressive strength, fineness of grinding, time of setting, blowing. The testing in compression of stones, bricks, and concrete. The testing of beams of joists, concrete plain and re-inforced. Impact and other tests (see the Author's Vol. II. on *Strength and Elasticity of Materials*).

Beams and Girders.—Bending moments, shear forces, and deflection of beams in general. Beams and girders fixed at the ends. Relations between curvature, slope, and deflection. Deflection and slope from bending moment diagrams. Resilience of beams. Combined bending and thrust (see the Author's Vols. II. and III.).

Struts.—The design of struts, Euler, Rankine, Gordon, and other formulæ. Effect of non-axial loading (see the Author's Vol. II.).

Shear.—Calculation of shear stresses and their distribution across any section of a loaded beam. Design of riveted joints in all classes of structural work (see the Author's Vol. III.).

Elementary Treatment of Earth and Water Pressures.—Determination of the thrust on the back of retaining walls, (a) not surcharged, (b) surcharged. Distribution of stress on the base of a retaining wall or dam. Depths of foundations for given loading (see the Author's Vols. II. and III.).

Re-inforced Concrete.—Moments of resistance of re-inforced concrete beams of rectangular and T section. Approximate theories only will be required. Re-inforced concrete columns, arrangement of armour.

Masonry and Metal Arches.—The candidates will be expected to have a more complete knowledge of the theory of masonry and brickwork arches, and of metal arches with three hinges. Graphical methods should be employed as much as possible (see the Author's Vol. III.).

Subject vii. (b).—Machines and Hydraulics.**STAGE 2.**

Candidates are understood to have an advanced knowledge of the subjects of the preceding course, being able to apply elementary mathematics to any practical problem. They must possess exact notions of a "rate" (as in velocity), or of an integral (as in finding an area, or finding work done under varying force), and show that they have understood what they are in the habit of doing with curves and columns of figures in practical problems such as :—Given the values of the force on a body at equal intervals of space, and the mass, to draw curves showing the velocity at each place and the time. For example, such problems as are sometimes given on the Bull engine.

Mechanism. Conversion of motion. Velocity ratios. Belts, ropes, chains, links. Sliding and rolling. Wheel trains. Chain gearing (see the Author's Vol. V. on *Theory of Machines*.)

A good general knowledge of the effects of friction. Screw friction. Rolling friction. Roller bearings (see the Author's Vol. I. on *Applied Mechanics*.)

Slipping of a belt. Length and strength of belts. Speed cones.

Applications of the principles of the dynamics of rotating bodies as in flywheels. Centrifugal force.

Effect of a blow. Reciprocating motions and vibration, linear and angular; measurement of torsional rigidity. Balancing of quick-moving machinery. Vibrations.

Changes of pressure and velocity along the stream lines in fluids; in the various parts of a centrifugal pump or turbine. Gauge notches for measuring water. Thomson's jet pump. Friction in pipes. Hydraulic propulsion. The effects of friction in pipes and passages of hydraulic machinery. Hydraulic and other lifts (see the Author's Vols. I., II., IV., and V.).

STAGE 3.

Candidates must possess a more advanced knowledge of the subjects of Stages 1 and 2.

Balancing in hydraulic and other lifts.

The results of experiments on friction at journals and pivots, in screws, and in pipes, &c., conveying fluids (see the Author's Vol. I.).

Mechanism. The kinematics and kinetics of machines. Velocity and acceleration diagrams and images. Linkages in general. Wheel teeth. Worm wheel teeth. Sliding and rolling contact. Mechanical integrators. Rolling cylinders and cones, and the most general motions of bodies. Slipping of belts. Centrifugal force in belts and rims of pulleys. Theory of flywheels (see the Author's Vol. V.).

Effect of a blow. Stoppage of water in a pipe. Vibration and the effect of friction in vibration. The balancing of machinery.

Whirling fluid. Centrifugal pumps. Fans and turbines. Hydraulic propulsion. The change in pressure and velocity along and across stream lines in water or air. "The rotation" in a fluid. Hydraulic transmission of power; ship resistance. Loss of head along a channel due to change of section (see the Author's Vols. I., IV., and V.).

City and Guilds of London Institute.

DEPARTMENT OF TECHNOLOGY.

TECHNOLOGICAL EXAMINATIONS.

46.—MECHANICAL ENGINEERING.*

HONOURS GRADE (Written Examination).

INSTRUCTIONS.

The Candidate for Honours must have previously passed in the Ordinary Grade, and is required to pass a Written and Practical Examination. He is requested to state, on the Yellow Form, whether he has elected to be examined in A, Machine Designing, or in B, Workshop Practice (*a*) *Fitting*, (*b*) *Turning*, (*c*) *Pattern-making*. Candidates in Machine Designing must forward their work to London not later than May 6th,† and in Workshop Practice not later than May 13th.†

The number of the question must be placed before the answer in the worked paper.

The Candidate is at liberty to use divided scales, compasses, set squares, calculators, slide rules, and mathematical tables.

Five marks extra will be awarded for every answer worked out with the slide rule, provided the method of working is explained.

The maximum number of marks obtainable is affixed to each question.

Three hours allowed for this paper.

The Candidate is not expected to answer more than *eight* of the following questions, which must be selected from *two* sections *only*.

* The questions in Sections A and C may be answered from Vols. I. to V. of my *Text-Book on Applied Mechanics*. Section B is printed in my *Text-Book on Steam and Steam Engines*. The Questions for the Ordinary Grade are printed at the end of my *Elementary Manual on Applied Mechanics*.

* See *Appendix C* for the latest Exam. Papers.

† These dates are only approximate, and subject to a slight alteration each year.

To obtain the certificate in Honours, the candidate must pass a Written and a Practical Examination, to be taken in the same year.

(1.) Written Examination.—In the Written Examination on *The Mechanics of Engineering*, candidates must select questions from not more than *two* of the following three divisions:—

(A.) The elasticity and strength of materials, including the more practical and elementary problems in compound stress. Tension, compression, and torsion. Combined bending and torsion. Combined thrust and bending. Riveted joints and the design of riveted work. Collapse. Behaviour of materials when tested. Ordinary limits of working stress (see the Author's Vol. II. on *Strength and Elasticity of Materials*).

(B.) The theory of the steam engine, including the thermodynamics of the action of steam. The solution of problems relating to the simpler valve gears. Governors and flywheels. The theory of gas engines, oil engines, and hot-air engines (see the Author's *Text-Book of Steam and the Steam Engine, including Turbines and Boilers*).

(C.) Hydraulics and hydraulic motor. Theory of flow from orifices. Flow in pipes. Water wheels, turbines and pumps. Construction and action of valves. Governors for hydraulic machinery. Hydraulic transmission of power. Hydraulic pressure engines. Lifts (see the Author's Vol. IV. on *Hydraulics*).

The Written Examination will be held in April of each year.

The Institution of Civil Engineers.

EXTRACTS FROM RULES AND SYLLABUS OF EXAMINATIONS FOR ELECTION OF ASSOCIATE MEMBERS.

Note.—Engineers who desire to enter for the A.M.Inst.C.E. examinations should write *at once* to the Secretary, Great George Street, Westminster, S.W., for the complete Rules, Syllabus, and Application Forms. They will find all the questions relating to the above mentioned subjects which have been set since these examinations commenced in 1897 in my **Text-Books**.

The following extracts are simply printed here to show how far my books upon *Applied Mechanics and Mechanical Engineering; Steam and Steam Engines, including Turbines and Boilers*; as well as *Magnetism and Electricity* (including Munro & Jamieson's *Pocket-Book of Electrical Rules and Tables*), together with my "*Correspondence System of Electrical and Mechanical Engineering Science, as taught by Exercises, Drawings, and Instructions*" cover the Scientific and Practical Knowledge demanded by the Institution, under Part II., **Section A** (1, 2, 3a, and 3b), as well as **Section B** under Group i., Theory of Heat Engines; Group ii., Hydraulics and Theory of Machines; and Group iii., Applications of Electricity.

Note for Students.—In the regulations for Students one of the subjects which may be selected is that of Elementary Mechanics of Solids and Fluids, for which see my *Elementary Applied Mechanics* when studying this subject. Also, see my *Elementary Magnetism and Electricity* book when reading that part of the Elementary Physics for admission of Students, or correspond with me *re* those two subjects.

PART II.*—Scientific Knowledge.

SECTION A.

1. **Mechanics** (one Paper, *time allowed, 3 hours*).
2. **Strength and Elasticity of Materials** (one Paper, *time allowed, 3 hours*).

* Candidates may offer themselves for examination in Sections A and B of Part II. together; or they may enter for Section A alone, and, if successful, may take Section B at a subsequent examination. In the latter case, however, such candidates will not be allowed to present themselves for examination in Section B unless or until they are actually occupied in work as pupils or assistants to practising Engineers. The Council may permit Candidates who have attempted the whole of Part II. at one examination, and have failed in Section B only, to complete their qualification by passing in that section at a subsequent examination, subject to their being then occupied as above stated.

3. *Either* (a) Theory of Structures,
 or (b) Theory of Electricity and Magnetism (one Paper,
time allowed, 3 hours).

SECTION B.

Two of the following nine subjects—not more than one from any group (one Paper in each subject taken, *time allowed, 3 hours for each Paper*):—

<i>Group i.</i>	<i>Group ii.</i>	<i>Group iii.</i>
Geodesy.	Hydraulics.	Geology and Mineralogy.
Theory of Heat Engines.	Theory of Machines.	Stability and Resistance of Ships.
Thermo- and Electro-Chemistry.	Metallurgy.	Applications of Electricity.

Mathematics.—The standard of Mathematics required for the Papers in Part II. of the examination is that of the mathematical portion of the Examination for the Admission of Students, though questions may be set involving the use of higher Mathematics.

The range of the examinations in the several subjects, in each of which a choice of questions will be allowed, is indicated generally hereunder:—

SECTION A.

1. Mechanics:—

Statics.—Forces acting on a rigid body; moments of forces, composition, and resolution of forces; couples, conditions of equilibrium, with application to loaded structures. The foregoing subjects to be treated both graphically and by aid of algebra and geometry.

Hydrostatics.—Pressure at any point in a gravitating liquid; centre of pressure on immersed plane areas; specific gravity.

Kinematics of Plane Motion.—Velocity and acceleration of a point; instantaneous centre of a moving body.

Kinetics of Plane Motion.—Force, mass, momentum, moment of momentum, work, energy, their relation and their measure; equations of motion of a particle; rectilinear motion under the action of gravity; falling bodies and motion on an inclined plane; motion in a circle; centres of mass and moments of inertia; rotation of a rigid body about a fixed axis; conservation of energy.

2. Strength and Elasticity of Materials:—

Physical properties and elastic constants of cast iron, wrought iron, steel, timber, stone, and cement; relation of stress and strain, limit of elasticity, yield-point, Young's modulus; coefficient of rigidity; extension and lateral contraction; resistance within the elastic limit in tension, compression, shear and torsion; thin shells; strength and deflection in simple cases of bending; beams of uniform resistance; suddenly applied loads.

Ultimate strength with different modes of loading; plasticity, working stress; phenomena in an ordinary tensile test; stress-strain diagram; elongation and contraction of area; effects of hardening, tempering and annealing; fatigue of metals; measurement of hardness.

Forms and arrangements of testing machines for tension, compression, torsion, and bending tests; instruments for measuring extension, compression, and twist; forms of test pieces and arrangements for holding them;

influence of form on strength and elongation; methods of ordinary commercial testing; percentage of elongation and contraction of area; test conditions in specifications for cast iron, mild steel, and cement.

3. (a) Theory of Structures :—

Graphic and analytic methods for the calculation of bending moments and of shearing forces, and of the stresses in individual members of framework structures loaded at the joints; plate and box girders; incomplete and redundant frames; stresses suddenly applied, and effects of impact; buckling of struts; effect of different end-fastenings on their resistance; combined strains; calculations connected with statically indeterminate problems, as beams supported at three points, &c.; travelling loads; riveted and pin-joint girders; rigid and hinged arches; strains due to weight of structures; theory of earth-pressure and of foundations; stability of masonry and brickwork structures.

3. (b) Theory of Electricity and Magnetism :—

Electrical and magnetic laws, units, standards, and measurements electrical and magnetic measuring instruments; the theory of the generation, storage, transformation, and distribution of electrical energy; continuous and alternating currents; arc and incandescent lamps; secondary cells.

SECTION B.

Group i. Theory of Heat Engines :—

Thermodynamic laws; internal and external work; graphical representation of changes in the condition of a fluid; theory of heat engines working with a perfect gas; air- and gas-engine cycles; reversibility, conditions necessary for maximum possible efficiency in any cycle; properties of steam; the Carnot and Clausius cycles; entropy and entropy-temperature diagrams, and their application in the study of heat engines; actual heat engine cycles and their thermodynamic losses; effects of clearance and throttling; initial condensation; testing of heat engines, and the apparatus employed; performances of typical engines of different classes; efficiency.

Group ii. Hydraulics :—

The laws of the flow of water by orifices, notches, and weirs; laws of fluid friction; steady flow in pipes or channels of uniform section; resistance of valves and bends; general phenomena of flow in rivers; methods of determining the discharge of streams; tidal action; generation and effect of waves; impulse and reaction of jets of water; transmission of energy by fluids; principles of machines acting by the weight, pressure, and kinetic energy of water; theory and structure of turbines and pumps.

Theory of Machines :—

Kinematics of machines; inversion of kinematic chains; virtual centres; belt, rope, chain, toothed and screw gearing; velocity, acceleration

and effort diagrams; inertia of reciprocating parts; elementary cases of balancing; governors and flywheels; friction and efficiency; strength and proportions of machine parts in simple cases.

Group iii. Applications of Electricity :—

Theory and design of continuous- and alternating-current generators and motors, synchronous and induction motors and static transformers; design of generating- and sub-stations and the principal plant required in them; the principal systems of distributing electrical energy, including the arrangement of mains and feeders; estimation of losses and of efficiency; principal systems of electric traction; construction and efficiency of the principal types of electric lamps.

28 Candidates should see, that all their "Forms" are duly completed and passed by the Council of the Institution of Civil Engineers, Great George Street, Westminster, S.W., before 1st January for the February Examination, and before the 1st September for the October Examination. Candidates should, therefore, apply to the Secretary for the "Forms," at least six months before these Examinations, to give them time to make due and proper Application, and to thoroughly *Revise* the subjects upon which they are to be examined with an experienced Guide and Tuition by Correspondence.

Examinations Abroad.—The papers of the *October Examination* only will be placed before accepted Candidates in India and the Colonies. To enable the Secretary to make arrangements for the Application Forms and Fees, &c., of these Candidates, their Forms, &c., must be in the Secretary's hands, before the 1st June preceding the October Examinations.

N.B.—See Appendix C for the latest Exam. Papers in Applied Mechanics.

APPENDIX B.

LECTURE II.—ORDINARY QUESTIONS.

1. Electric current is supplied to a certain motor plant at 220 volts, and 150 amperes are taken. What H.P. does this represent? How much would it cost if used for an average of 6.5 hours per day for a whole year of 313 days? The power is supplied at 2.24d. per H.P.-hour (i.e., 3d. per B.T.U.). (B. of E., Adv., 1903.)

LECTURE III.—ORDINARY QUESTIONS.

1. The figure shows a bent lever, AOB. The fulcrum at O is in a loose cylindric bearing 4 ins. diameter, coefficient of friction 0.3. AO is 12 ins., BO is 24 ins.; the force Q of 1,000 lbs. acts at A. What force P acting at



B will just overcome Q and the friction? Find also the line of action and magnitude of the force acting upon the lever at O.

(B. of E., S. 2 and 3, 1904.)

2. The figure shows the skeleton mechanism of a direct-acting steam engine, with a connecting-rod 4 cranks long, stroke 26 inches. In the

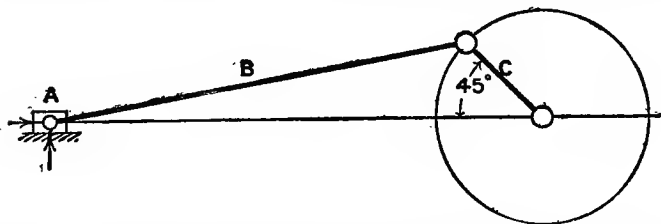


figure the crank has turned through an angle of 45° from the dead centre in a clockwise direction. The diameter of the piston is 13 inches, and the effective steam pressure upon it at this instant is 100 lbs. per square inch;

the crosshead pin is $3\frac{1}{2}$ inches in diameter, the crank pin is $4\frac{1}{2}$ inches in diameter, and the crank shaft 5 inches in diameter. The coefficient of friction between the crosshead and guide bar is 0.088, and the coefficient of friction for the three pins is 0.052. Find graphically, or otherwise, the turning moment exerted on the crank shaft. (B. of E., S. 2 and 3, 1905.)

LECTURE IV.—ORDINARY QUESTIONS.

1. In a crane an effort of 122 lbs. just raises a load of 3,265 lbs. What is the mechanical advantage? If the efficiency be 60 per cent. what is the velocity ratio? Would this crane overhaul? (B. of E., Adv., 1903.)

2. Answer only *one* of the following, (a), (b), or (c):—(a) Describe how you would proceed to determine experimentally for a screw-jack (1) its velocity ratio, (2) its mechanical efficiency at a number of different loads; (b) describe, with a sketch of the apparatus, how you would experimentally determine the law connecting the twisting moment and the angle of twist for a given piece of steel wire; (c) describe how you would prepare and test for tensile stress a specimen, or briquette, as it is commonly called, of neat Portland cement. (B. of E., S. 2, 1905.)

LECTURE VI —ORDINARY QUESTIONS.

1. Experiments made by Froude to determine the skin resistance of planks in water give the following results:—

Speed in feet per minute = V,	200	400	600	800
Total resistance per 100 sq. ft. in lbs. = R,	3.28	11.7	24.6	41.7

Test whether the relation between R and V can be expressed by a law of the type R varies as V^n , and, if so, find the values of f and n in the formula $R = fSV^n$, in which S = wetted surface in square feet.

(C. & G., 1903, H., Sec. C.)

2. The friction of a thin plate when moved edgewise through water is found by experiment to be $\frac{1}{4}$ lb. per square foot of surface in contact with the water, when the velocity of rubbing is 600 feet per minute, and that it varies as the square of the velocity of rubbing. How many foot-lbs. of work per minute will be expended in overcoming the skin friction in the case of a ship steaming at $18\frac{1}{2}$ knots, if the immersed surface of the ship when floating at her load line is 27,620 square feet? If this skin friction is 70 per cent. of the total resistances encountered by the ship, what is the total horse-power usefully expended in propelling the ship?

(B. of E., S. 2, 1906.)

LECTURE X.—ORDINARY QUESTIONS.

1. Part of a machine weighing 1 ton is moving northwards at 60 feet per second. At the end of 0.05 second it is found to be moving to the east at 20 feet per second. What is the average force (find magnitude and direction)

acting upon it during the interval 0.05 second? What is meant by "averages" in such a case? What is meant by *force* by people who have to make exact calculations? (B. of E., S. 3, 1904.)

2. The angular position D of a rocking shaft at any time t is measured from a fixed position. Successive positions at intervals of $\frac{1}{16}$ second have been determined as follows:—

Time t , seconds, .	0.00	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18
Position D , radians,	0.106	0.208	0.337	0.487	0.651	0.819	0.978	1.111	1.201	1.222

Find the change of angular position during the first interval from $t = 0.00$ to $t = 0.02$. Calculate the mean angular velocity during this interval in radians per second, and, on a time base, set this up as an ordinate at the middle of the interval. Repeat this for the other intervals, tabulating the results, and drawing the curve showing approximately angular velocity and time. In the same way find a curve showing angular acceleration and time. Read off the angular acceleration in radians per second per second when $t = 0.075$ second. A wheel keyed to the shaft weighs 720 lbs., and has a radius of gyration of 1.5 feet. What is the torque tending to fracture the shaft when $t = 0.16$ second? (B. of E., S. 2 and 3, 1905.)

3. Three adjacent positions, G_1 , G_2 , G_3 of the centre of mass G of a balance weight, at intervals of $\frac{1}{16}$ second, have been found by geometrical construction. Referred to perpendicular axes, these positions are measured in feet as follows:—

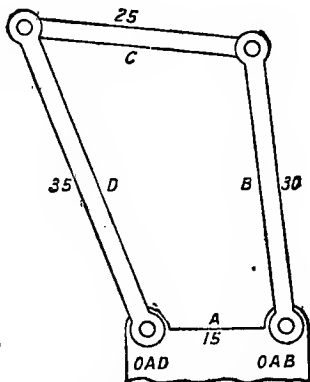
	x	y
G_1 . . .	0.167	0.078
G_2 . . .	0.352	0.146
G_3 . . .	0.487	0.242

Find approximately, by taking first and second differences, the x and y components of the velocity and acceleration of G when in the position G_2 . Find also the resultant velocity and resultant acceleration for this position. The mass of the balance weight was 351 lbs. Find the magnitude and direction of the force corresponding to the acceleration for the position G_2 . Plot the points on squared paper, and on the diagram exhibit the velocity, acceleration, and force as vectors. (B. of E., S. 2 and 3, 1906.)

4. In the four-bar mechanism shown in the sketch, the bar A is a fixed bar; the bars B and D rotate about the fixed centres OAB and OAD , and they are coupled together at their outer ends by the bar C ; the bar B revolves in clockwise direction with uniform velocity round its fixed axis OAB at 50 revolutions per minute. Find, in any way you please, the

position of the bar D for 12 equidistant positions of the bar B during one complete revolution, and fill in a table similar to the one given below:—

Angle turned through by the bar B.	Angle turned through by the bar D.	Mean Angular Velocity of the bar D in Radians per Second during each Interval.
Degrees.	Degrees.	
0		
30		
60		
&c.		



The lengths of the bars are 15, 30, 25, and 35 inches respectively.

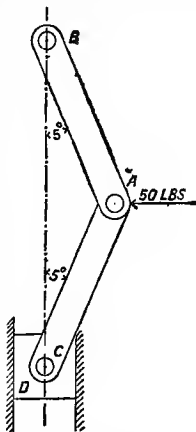
Plot a curve showing the relation between the angular velocities of B and D. Find the values of the maximum and minimum accelerations of the link D. (B. of E., S. 2 and 3, 1907.)

5. The sketch shows the mechanism known as a knuckle-joint, which consists of three moving links and a fourth fixed link—viz., the frame of the machine. Determine, for the position of the mechanism shown in the sketch, the virtual or instantaneous centre of rotation of each of the three links AB, AC, and D in regard to the fixed link. A force of 50 lbs. is applied at the joint A in a direction perpendicular to BC. Determine the vertical thrust delivered by the block D, neglecting friction.

(B. of E., S. 2, 1908.)

6. A cyclist is riding due west at a speed of 12 miles per hour, and the wind is at the time blowing from the south-east with a speed of $5\frac{1}{2}$ miles per hour. If the cyclist carries a small flag, in what direction will this flag fly? At what speed would the cyclist require to ride if the flag is to fly due north?

(B. of E., S. 2, 1908.)



LECTURE XI.—ORDINARY QUESTIONS.

1. A projectile has kinetic energy = 1,670,000 foot-lbs. at a velocity of 3,000 feet per second. Later on its velocity is only 2,000 feet per second; how much kinetic energy has it lost? What is the cause of this loss of energy? Calculate the kinetic energy of rotation of the projectile if its weight is 12 lbs., and

its radius of gyration is 0.75 inch, and its speed of rotation is 500 revolutions per second. (B. of E., Adv., 1903.)

2. A train weighing 250 tons is moving at 40 miles per hour, and it is stopped in 10 seconds. What is the average force during these 10 seconds causing this stoppage? Define what is meant by force by people who have to make exact calculations. (B. of E., S. 2, 1904.)

3. A body whose weight is 350 lbs. is being acted upon by a variable lifting force F lbs. when it is at the height x feet from its position of rest. The mechanism is such that F depends upon x in the following way, but the body will stop rising before the greatest x of the table is reached. Where will it stop?

x . . .	0	15	25	50	70	100	125	150	180	210
F . . .	350	525	516	490	425	300	210	160	110	90

Where does its velocity cease to increase and begin to diminish? What work is done by F in lifting the body 100 feet? If the weight of the body is 350 lbs., how much of its energy is potential and how much kinetic when it has been lifted 100 feet? (B. of E., S. 2 and 3, 1904.)

4. A truck, weighing 5 tons without its wheels, rests on four wheels, which are circular discs, 40 inches in diameter, each weighing $\frac{1}{2}$ ton, and moves down an incline of 1 in 60. Find the velocity of the truck in feet per second after moving 100 feet from rest, if the resistance due to friction is 1 per cent. of the weight. What percentage of the original potential energy has been wasted in friction? (B. of E., S. 3, 1905.)

5. A bicycle and its rider weigh 200 lbs.; the distance travelled for each turn of the pedal is equal to the circumference of a wheel having a diameter of 80 inches; neglecting frictional and other resistances, how many foot-lbs. of work will the cyclist do per revolution of the pedal in ascending a hill with a gradient of 4 in 100. If the resistance due to friction, air resistance, &c., is $2\frac{1}{2}$ lbs., how many foot-lbs. of work per minute is this cyclist doing when he maintains a steady speed of 6 miles per hour up the hill? (B. of E., S. 2, 1907.)

6. A solid cast-iron disc, 40 inches in diameter and 8 inches thick, is rotating at a uniform speed of 240 revolutions per minute. If the air frictional resistance is assumed to be equal to KV^2 lbs. per square foot, where V is the linear velocity of any point, obtain an expression for the horse-power required to keep the disc in rotation. (B. of E., S. 2, 1908.)

LECTURE XII.—ORDINARY QUESTIONS.

1. A flywheel weighs 5 tons and has a radius of gyration of 6 feet. What is its moment of inertia? It is at the end of a shaft 10 feet long, the other end of which is fixed. It is found that a torque of 200,000 lb.-feet is sufficient to turn the wheel 1° . The wheel is twisted slightly and then released; find the time of a complete vibration. How many vibrations per minute would it make? (B. of E., H., Part I., 1903.)

2. A flywheel and its shaft weigh 24,000 lbs., its bearings which are slack are 9 inches diameter. If the coefficient of friction is 0.07, how many foot-lbs. of work are wasted in overcoming friction in one revolution? If

the mean radius (or rather the radius of gyration) is 10 feet, what is the kinetic energy when the speed is 75 revolutions per minute? If it is suddenly disconnected from its engine at this speed, in how many revolutions will it come to rest? What is the average speed in coming to rest? In how many minutes will it come to rest? (B. of E., S. 2, 1904.)

3. A motor car moves in a horizontal circle of 300 feet radius. The track makes sideways an angle of 10° with the horizontal plane. A plumb line on the car makes an angle of 12° , with what would be a vertical line on the car if it were at rest on a horizontal plane; what is the speed of the car? If the car is just not side-slipping, what is the coefficient of friction? (B. of E., S. 2 and 3, 1904.)

4. A cast-iron flywheel is 10 feet in diameter, and the cast iron of which it is made weighs 0.26 lb. per cubic inch. How many revolutions per minute can this flywheel be allowed to make if the tensile stress on the rim due to the centrifugal force is not to exceed 2 tons per square inch? Prove the truth of any formula you may employ. (B. of E., S. 2, 1905.)

5. A flywheel is supported on an axle $2\frac{1}{2}$ inches in diameter, and is rotated by a cord, which is wound round the axle and carries a weight. It is found by experiment that a weight of 5 lbs. on the cord is just sufficient to overcome the friction and maintain steady motion. A load of 25 lbs. is attached to the cord, and 3 seconds after starting from rest it is found that the weight has descended 5 feet. Find the moment of inertia of this wheel in engineers' units. If the wheel is a circular disc 3 feet in diameter, what is its weight (The thickness of the cord may be neglected.) (B. of E., S. 3, 1905.)

6. A flywheel mounted on a horizontal spindle in bearings is rotated by winding a cord on the spindle, attaching a weight to the cord, and allowing the weight to fall to the ground. In an actual experiment the falling weight was 21 lbs.; the total height of fall, 5 feet; the height of fall of the weight for one revolution of the spindle was 5.05 inches; the time taken by the weight from starting from rest to reach the floor was 7.6 seconds, the whole time of rotation of the flywheel starting from rest was 70.25 seconds, and the total number of rotations of the flywheel was 109.9. Find (a) the energy in inch-lbs. in the falling weight at the instant of striking the floor; (b) the energy in inch-lbs. per revolution lost in friction in the bearing of the spindle; (c) the moment of inertia of the flywheel. (B. of E., S. 2, 1906.)

7. A punching machine needs 4 H.P., a flywheel upon the machine fluctuates in speed between 100 and 110 revolutions per minute; a hole is punched every three seconds, and this requires five-sixths of the total energy given to the machine during the three seconds. Find the M and the I of this flywheel. "M" is the kinetic energy of the wheel at 1 revolution per minute. (B. of E., S. 2, 1907.)

8. The rim of a cast-iron pulley has a mean radius of 12 inches, the rim is 6 inches broad and $\frac{1}{2}$ inch thick, and the pulley revolves at the rate of 150 revolutions per minute; what is the centrifugal force on the pulley rim per 1-inch length of rim? (1 cubic inch of cast-iron weighs 0.26 lb.) What is the tensile stress per square inch in the pulley rim under these conditions? What is the limiting speed of rotation for this pulley if the tenacity of cast iron is 12.5 tons per square inch. (B. of E., S. 2, 1907.)

LECTURE I.—I.C.E. QUESTIONS.

1. Distinguish between the mass, weight, density, and specific gravity of a substance. (I.C.E., Oct., 1905.)

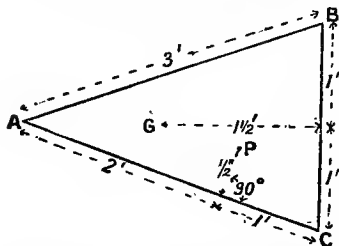
LECTURE III.—I.C.E. QUESTIONS.

1. Sketch the common steelyard. In a given steelyard the lever weighs 6 lbs., and its centre of gravity is 2 inches to the left of the axis; the hooks and links weigh 2 lbs. and are suspended from a knife-edge 3 inches to the left of the axis, and the farthest graduation is 30 inches to the right of the axis. The steelyard weighs up to 60 lbs. load. Find the weight of the rider, and the position of zero graduation. (I.C.E., Feb., 1905.)

2. In a lever safety valve the mean diameter of the valve is 2 inches, and its weight is 2 lbs.; the distance of the centre of the valve from the fulcrum is 2 inches, the weight of the lever is 10 lbs., and the distance of its centre of gravity from the fulcrum is 9 inches. Find where a weight of 20 lbs. must be hung in order that steam may blow off at 100 lbs. per square inch. (I.C.E., Oct., 1905.)

3. A barge, 70 feet long, having its centre of gravity equidistant from its ends is being towed by a rope which is fastened to a point 20 feet from the bow, and makes an angle of 15° with the side of the canal. If the pull in the rope is 90 lbs., find (i.) the effective force urging the barge forward; (ii.) the moment of the rudder, in pound-foot units, required to keep the barge parallel to the tow-path. (I.C.E., Oct., 1906.)

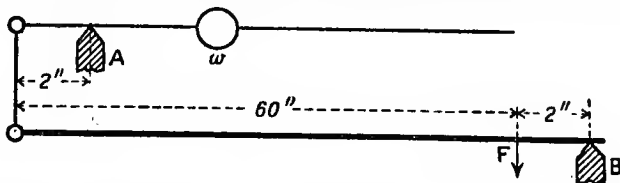
4. A rigid platform, weighing 2 tons, is supported by rigid struts at A, B, and C, as shown by the sketch. Its centre of gravity is at G, and it is loaded with 4 tons at P. Find the pressure on the struts. (I.C.E., Oct., 1907.)



5. Prove that any system of forces acting as a rigid body in one plane may be replaced by a single force acting at a given point and a couple. ABCDEF are the corners, taken in order, of a regular hexagon, of 1-foot side. A force of 300 lbs. acts from A towards F, a

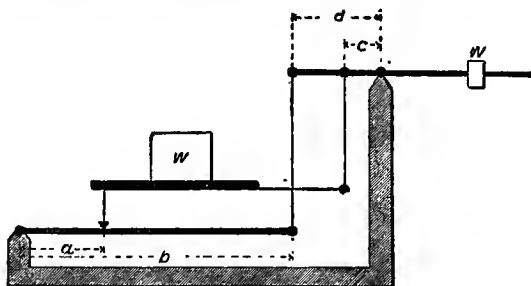
force of 400 lbs. acts from F towards E, and a force of 500 lbs. acts from D towards E. Find the magnitude and direction of the equivalent force through B, and the magnitude and direction of the couple. (I.C.E., Feb., 1908.)

6. A lever system is shown diagrammatically in the figure, A and B being fixed fulcrums. The system is balanced when the weight w , 100 lbs.,



is 4 inches to the right of A. Find the position of w when a force, F , of 20,000 lbs. is applied as shown. (I.C.E., Feb., 1908.)

7. The sketch shows diagrammatically the arrangement of levers in a platform weigh-bridge. Investigate the connection between the lengths of



a , b , c , and d , in order that the balance may be independent of the position of the weight W on the platform. (I.C.E., Oct., 1908.)

LECTURE IV.—I.C.E. QUESTIONS.

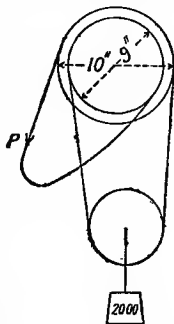
1. Explain what is meant by "velocity ratio," "mechanical advantage," and "efficiency" as applied to rope tackle; and state the experimental law which is generally found to exist between the force applied and the load lifted. In a Weston pulley block the diameters of the two pulleys are 8 inches and $7\frac{1}{2}$ inches respectively, and it is found that a pull of 7 lbs. will just raise a load of 24 lbs., and a pull of 25 lbs. a load of 240 lbs. Find the probable pull required to lift 600 lbs. and the efficiency under this load. *Ans.* Pull required to lift 600 lbs. = 55 lbs., and the efficiency under this load is 34.1 per cent. (I.C.E., Oct., 1903.)

2. A crane, when tested, is found to require a force of 8.5 lbs. at the handle to lift a weight of 100 lbs.; and a force of 21.4 lbs. to lift a weight of 400 lbs. If the relation between the force supplied and the weight lifted follows the straight line law, find the equation giving the relationship between them. Moreover, if the velocity ratio of the points of application of the applied force and the weight be 30, find the efficiency of the crane when lifting a load of 800 lbs. *Ans.* Equation giving the relationship between the force applied and the weight lifted is $F = .043W + 4.2$; efficiency of crane when lifting a load of 800 lbs. is 69 per cent. (I.C.E., Feb., 1904.)

3. Sketch a seven-fold purchase tackle, and, denoting the velocity of the running block by unity, find the velocity of each ply of the rope. (I.C.E., Feb., 1904.)

4. The diameters of the two pulleys of a Weston pulley-block are 10 inches and 9 inches. Find the pull P required to raise 2,000 lbs., if the efficiency for that load is 65 per cent. (I.C.E., Oct., 1906.)

5. It was found impossible to adjust the position of the jockey-weight on the steelyard of a certain ordinary platform balance so that the steelyard



would float midway between the stops when there was a load on the platform. The slightest movement of the jockey-weight caused the steelyard to run to one or other of the stops when the jockey-weight just about indicated the load on the platform. What is wrong with the steelyard, and how would you remedy it? (I.C.E., Oct., 1906.)

6. Sketch and describe any one form of the differential pulley-block. How would you proceed to measure experimentally the efficiency of the blocks? (I.C.E., Oct., 1906.)

7. In a 30-ton testing machine the line of action of the stress in the test-piece is 3 inches to the left of the knife-edge on which the steelyard rests, and the machine is in equilibrium when the jockey-weight is 6 inches to the left of the knife-edge. If the jockey weighs 1 ton, construct a scale showing the pull in the standard test-piece of one square-inch section, for every position of the jockey referred to the knife-edge. (I.C.E., Feb., 1907.)

8. Sketch a differential pulley-block, such as the "Weston," based on the principle of the differential axle. In such a pulley-block the smaller diameter is $\frac{1}{2}$ of the large diameter. Neglecting friction, find the force necessary to raise a weight of 3,000 lbs. (I.C.E., Feb., 1908.)

LECTURE V.—I.C.E. QUESTIONS.

1. A gate 6 feet high and 4 feet wide, weighing 100 lbs., hangs from a rail by two wheels at its upper corners. The left-hand wheel, having seized, skids along the rail with a coefficient of friction of $\frac{1}{2}$. The other wheel is frictionless. Find the horizontal force that will push the gate steadily along from left to right, if applied 2 feet below the rail. You may solve either analytically or by the force and link polygon. *Ans.* $P = 20$ lbs. (I.C.E., Oct., 1904.)

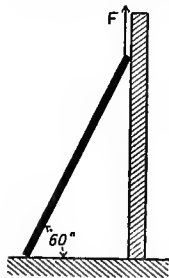
2. A ladder, whose centre of gravity is at the middle of its length, rests on the ground and against a vertical wall; the coefficients of friction of the ladder against both being $\frac{1}{2}$. Find the ladder's inclination to the ground when just on the point of slipping. *Ans.* Angle of inclination to horizontal = 62° . (I.C.E., Oct., 1904.)

3. How does the coefficient of dry friction vary with the speed of rubbing? Does this explain why it is better not to skid the wheels when bringing a train to rest? (I.C.E., Oct., 1904.)

4. State the laws of solid friction, and explain what is meant by the angle of friction. A rectangular block, weighing 50 lbs., is set up on a rough horizontal plane, its base being 8 inches square. A force of 10 lbs., applied horizontally parallel to one of the sides, is just sufficient to make it slide if applied below a certain point, but causes it to topple over if applied above that point. Find the height of this point, and the value of the coefficient of friction between the block and table. (I.C.E., Feb., 1905.)

5. A uniform beam of weight 1,500 lbs. is inclined at an angle of 60° to the horizontal, and its ends rest on the ground and against a wall, as shown in the figure. If the coefficient of friction for both the ground and the wall be 0.15, find the force F necessary to prevent the beam from slipping. (I.C.E., Feb., 1908.)

6. Distinguish between "static coefficient of friction" and "kinetic coefficient of friction." How does the latter vary with speed? Describe



the difference there is in frictional resistance between lubricated surfaces (as in a bearing) at low and high speeds, the oil supply being copious in both cases. (I.C.E., *Feb.*, 1908.)

LECTURE VI.—I.C.E. QUESTIONS.

1. Sketch the arrangement you would employ for lubricating the bearings of (a) a railway-carriage axle, (b) a crank-shaft bearing of a reciprocating engine, and (c) the bearing of a dynamo running at 1,500 revolutions per minute. (I.C.E., *Oct.*, 1908.)

LECTURE VII.—I.C.E. QUESTIONS.

1. Sketch a good form of a ball-bearing suitable for the front wheel of a bicycle. (I.C.E., *Feb.*, 1907.)

LECTURE VIII.—I.C.E. QUESTIONS.

1. With the aid of sketches describe one form of dynamometer for measuring the turning moment exerted on a revolving shaft and describe the process of measurement. (I.C.E., *Oct.*, 1904.)

2. What are the causes producing friction in a double-acting vertical steam engine, and to what extent does this friction vary with the load on the engine? If the brake efficiency of a certain engine is 92 per cent. at full load, what would approximately be its brake efficiency at $\frac{1}{2}$ load? (I.C.E., *Feb.*, 1906.)

3. A rope brake, placed on the flywheel of a steam engine, carries a load of 250 lbs. hanging freely, and when the wheel makes 240 revolutions per minute the tension on the fixed end of the brake rope is 4 lbs. The diameter of the flywheel is 5 feet and the girth of the rope is 2.5 inches. Calculate the brake horse-power of the engine and the heat-units produced by the action of the brake per second. Make a sketch of a suitable arrangement of the brake. (I.C.E., *Feb.*, 1907.)

LECTURE IX.—I.C.E. QUESTIONS.

1. A plane inclined at 20° to the horizontal carries a load of 1,000 lbs., and the angle of friction between the load and plane is 10° . Obtain the least force in magnitude and direction which is necessary to pull the load up the plane. *Ans.* Least force is 500 lbs. at 10° to the plane. (I.C.E., *Oct.*, 1903.)

2. Find the efficiency of a screw-jack of $\frac{3}{8}$ -inch pitch and mean diameter of thread 2 inches, the coefficient of friction of the screw thread being $\frac{1}{4}$. *Ans.* Efficiency of screw-jack = 19.3 per cent. (I.C.E., *Feb.*, 1904.)

3. An iron wedge used for splitting a tree is struck so as to be subject to a vertical force of 4 tons. The taper of the wedge is 1 inch per foot, and the coefficient of friction against the tree is 0.1. Find the horizontal splitting force, and find the efficiency. (I.C.E., *Oct.*, 1904.)

4. Ten railway trucks, each weighing 10 tons, have wheels 3 feet in diameter, with axles 4 inches in diameter. If the coefficient of friction of the journals is 0.005, find the horse-power required to draw the trucks up an incline of 1 in 300 at 30 miles per hour. (I.C.E., *Oct.*, 1905.)

5. In a screw-jack the pitch of the screw is $\frac{1}{2}$ inch, and the mean radius of the screw is 2 inches. If the coefficient of friction is 0.1, what is the efficiency of the jack? (I.C.E., *Feb.*, 1906.)

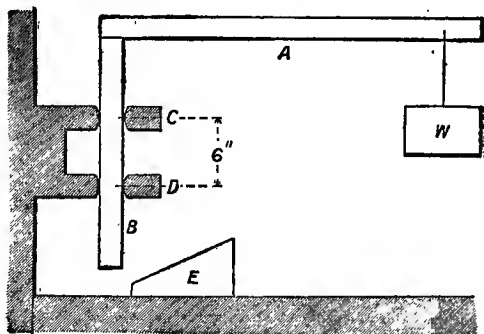
6. A screw-jack has a mean diameter of $2\frac{1}{2}$ inches and a pitch of $\frac{3}{8}$ inch. It is turned by applying a force P tangentially at the end of a horizontal lever 25 inches from the centre of the screw. Find (i.) the magnitude of P when the jack is lifting 3 tons, neglecting friction, (ii.) the coefficient of friction between the threads that just suffices to prevent the screw running back under the 3-ton load. (I.C.E., Feb., 1907.)

7. The diameter of the screw in a screw-jack is 2 inches, and the pitch of the screw is $\frac{1}{4}$ inch. Find an expression giving the relation between the torque exerted on the screw, and the load supported by the jack, (i.) neglecting friction, (ii.) including the friction of the screw and assuming ϕ for the limiting angle of resistance between the screw and the nut. (I.C.E., Feb., 1907.)

8. Describe the form of gearing known as worm and worm-wheel, and illustrate your remarks by means of hand-sketches of a double-threaded worm. Give one example of the practical use of such gears, and point out the advantages and disadvantages as compared with ordinary toothed gearing. (I.C.E., Oct., 1907.)

9. A heavy lathe is driven by an electric motor by means of worm gearing. Describe and give hand sketches of the arrangement of the worm gearing you would adopt. What method of lubrication would you provide? The speed of the motor may be taken at 500 revolutions per minute, that of the worm as 20, and the power to be transmitted as 40 H.P. (I.C.E., Feb., 1908.)

10. A weight W is supported by the L-shaped link AB which can slide vertically in the ring guides C and D , the fit in the rings being a loose one, and the diameter of B being 1 inch. Find the position of W along the



horizontal arm at which AB will begin to slide in the guides. Coefficient of friction 0.2. Neglect the weight of the link AB . Would it or would it not be possible to raise W by means of a wedge E placed under B in the direction indicated in the figure? Give reasons for your answer. (I.C.E., Oct., 1908.)

LECTURE X.—I.C.E. QUESTIONS.

1. In an electric railway the average distance between stations is $\frac{1}{2}$ mile, the running time from start to stop $1\frac{1}{2}$ minutes, and the constant speed between the end of acceleration and beginning of retardation 25 miles

an hour. If the acceleration and retardation be taken as uniform and numerically equal, find their values; and if the weight of the train be 150 tons and the frictional resistance 11 lbs. per ton, find the tractive force necessary to start on the level. (I.C.E., Oct., 1903.)

2. The acceleration of a train running on the level is found by hanging a short pendulum from the roof of a carriage and noticing the angle which the pendulum makes with the vertical. In one experiment the angle of inclination was 5° ; estimate the acceleration of the train in feet per second per second and miles per hour per hour. (I.C.E., Feb., 1904.)

3. To a passenger in a train moving at the rate of 40 miles an hour, the rain appears to be rushing downward and towards him at an angle of 20° with the horizontal. If the rain is actually falling in a vertical direction, find the velocity of the raindrops in feet per second. (I.C.E., Feb., 1904.)

4. Two weights, one of 2 lbs. and the other of 1 lb. are connected by a massless string which passes over a smooth peg. Find the tension in the string and the distance moved through by either weight, from rest, in 2 seconds. (I.C.E., Feb., 1904.)

5. Explain how, in the ordinary direct-acting engine, the velocity ratio between the piston and crank-pin at any point of the stroke may be obtained; and sketch a curve of velocity of the piston on a piston base, the crank-pin moving uniformly. Show how, from the curve so obtained, the acceleration of the piston at any point of the stroke may be deduced. (I.C.E., Feb., 1904.)

6. The speed of a motor car is determined by observing the times of passing a number of marks placed 500 feet apart. The time of traversing the distance between the first and second posts was 20 seconds, and between the second and third 19 seconds. If the acceleration of the car is constant, find its magnitude in feet per second per second, and also the velocity in miles per hour at the instant it passes the first post. (I.C.E., Feb., 1905.)

7. In a bicycle, the length of the cranks is 7 inches, the diameter of the back wheel is 28 inches, and the gearing is such that the wheel rotates $2\frac{1}{2}$ times as fast as the pedals. If the weight of the cyclist and machine together is 160 lbs., estimate the force which will have to be applied to the pedal to increase the speed uniformly from 4 to 12 miles an hour in 20 seconds—frictional losses being neglected. (I.C.E., Feb., 1905.)

8. A locomotive draws a train of 100 tons with a uniform acceleration such that a speed of 60 miles per hour is attained in four minutes on the level. If the frictional resistances are 10 lbs. per ton and the resistance of the air, which varies as the square of the speed, is 120 lbs. at 20 miles per hour, find the pull exerted by the locomotive at 30 and at 60 miles per hour. (I.C.E., Feb., 1906.)

9. Define the term "virtual" or "instantaneous centre," and explain how such centres can be used to determine the motion of link work. Illustrate your remarks by means of an example such as that of an ordinary crank and connecting-rod. (I.C.E., Feb., 1906.)

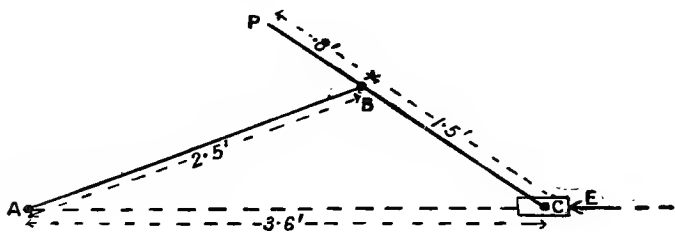
10. A man, standing on a train which is moving with a speed of 36 miles per hour, shoots at an object running away from the railway at right angles at a speed of 12 miles per hour. If the bullet, which is supposed to move in a horizontal straight line, has a velocity of 880 feet per second, and if the line connecting man and object makes an angle of 45° with the train when he fires, find at what angle to the train he must aim in order to hit the object. (I.C.E., Oct., 1906.)

11. Prove the formula giving the radial acceleration of a particle traversing a circle of radius R feet with a uniform speed of V feet per

second, and thence find the speed of horizontal rotation of a heavy ball swinging by a string, when the string makes an angle of 30° with the vertical. (I.C.E., Feb., 1907.)

12. A B is a link, 3 inches long, rotating in a vertical plane about A as centre. C B is a second link, 15 inches long, pivoted to the first at B. The end C moves to and fro along the horizontal line through A. Find (i.) the instantaneous centre of motion of C B when the angle C A B is 30° , (ii.) the relative velocities of C and B in that position. Hence show clearly how to plot a curve of velocities for C. (I.C.E., Feb., 1907.)

13. Define "instantaneous centre." In the mechanism shown by the



sketch a force E of 2,000 lbs. acts on the slider C and moves it at the rate of 1 foot per second. Find the velocity of P in magnitude and direction. What weight hung vertically at P will balance the force E? (I.C.E., Oct., 1907.)

14. Find the force, parallel to the incline, which will give an acceleration of $\frac{1}{10}$ foot per second per second to a load of 30 tons running up an incline of 1 in 50, the resistance due to friction being 40 lbs. per ton. (I.C.E., Feb., 1908.)

15. Find the position of the instantaneous centre of a motor-car wheel 30 inches in diameter when skidding, the car travelling at 10 miles per hour and the wheel revolving at a speed which would correspond to 4 miles per hour without slip. A bicycle with 30-inch wheels geared to 72 inches is held upright with its cranks vertical. A cord is fastened to the end of the *down* crank, and a pull is applied to the cord in the horizontal backward direction. In which direction will the bicycle move? If the crank were gradually lengthened from 6 inches to 6 feet (the bicycle being supported on a scaffold to enable the crank to clear the ground), and the string always applied as before at its end, would the direction of motion be always the same? Give reasons for your answers. (I.C.E., Oct., 1908.)

LECTURE XI.—I.C.E. QUESTIONS.

1. The mass of a flywheel may be assumed concentrated in the rim. If the diameter is 7 feet and the weight $2\frac{1}{2}$ tons, estimate its kinetic energy when running at 250 revolutions per minute. Moreover, if the shaft be 6 inches in diameter and the coefficient of friction of the shaft in the bearings be 0.09, estimate the number of revolutions the flywheel will make before coming to rest. (I.C.E., Oct., 1903.)

2. State the second law of motion. A cage weighing 1,000 lbs. is being lowered down a mine by a cable. Find the tension in the cable (i.) when the speed is increasing at the rate of 5 feet per second per second; (ii.) when the speed is uniform; (iii.) when the speed is diminishing at the rate of 5 feet per second per second. The weight of the cable itself may be neglected. (I.C.E., Oct., 1903.)

3. With an automatic vacuum brake a train weighing 170 tons and going at 60 miles an hour on a down gradient of 1 in 100 was pulled up in a distance of 596 yards. Estimate the total resistance in lbs. per ton, and if the retardation is uniform find the time taken to bring the train to rest. (I.C.E., Oct., 1903.)

4. If it take 600 useful H.P. to draw a train of 335 tons up a gradient of 1 in 264 at a uniform speed of 40 miles an hour, estimate the resistance per ton other than that due to ascending against gravity, and deduce the uniform speed on the level when developing the above power. (I.C.E., Feb., 1904.)

5. In a steam hammer the diameter of the piston is 36 inches, the total weight of the hammer and piston is 20 tons, and the effective steam pressure is 40 lbs. per square inch. Find the acceleration with which the hammer descends, and its velocity after descending through a distance of 4 feet; and, if the hammer then come into contact with the iron and compress it through a distance of 1 inch, find the mean force of compression. (I.C.E., Feb., 1904.)

6. A solid circular cast-iron disc, 20 inches in diameter and 2 inches thick (weighing 0.25 lb. per cubic inch) is mounted on ball bearings. A weight of 10 lbs. is suspended by means of a string wound round the axle, which is 3 inches in diameter; and the weight is released and disconnected after falling 10 feet. Neglecting friction, find the kinetic energy stored in the wheel, and the revolutions per minute which the wheel is making when the weight is disconnected; and also the time it would continue to run against a tangential resistance of $\frac{1}{2}$ lb. applied at the circumference of the axle. (I.C.E., Feb., 1904.)

7. A waggon begins to ascend a gradient of 1 in 60 at 15 miles an hour. The resistance is 20 lbs. per ton. How far will it run up? What will be its speed when it gets to the bottom again? (I.C.E., Oct., 1904.)

8. A cylinder rolls down a rough inclined plane. Form the equations of motion. Determine the acceleration. Find the coefficient of friction which will just prevent sliding. (I.C.E., Oct., 1904.)

9. A bucket weighing 50 lbs. is hauled up a shaft by a rope wound round an axle, the weight of which is 70 lbs. If the axle is suddenly released when the bucket is 20 yards from the bottom of the shaft, find the velocity with which the bucket reaches the bottom, neglecting friction and the inertia of the rope. (I.C.E., Oct., 1905.)

10. A train is travelling at a uniform speed on the level. The weight of the brake-van at the rear of the train is 10 tons and the weight of the remaining part of the train is 90 tons. If the brakes are applied to the brake-van, what will be the force on the brake-van couplings, the coefficient of friction of wheels on the rails being 0.1? (I.C.E., Feb., 1906.)

11. The final blow on a 9-inch diameter pile, 20 feet long, from a weight of 1,800 lbs., dropping 25 feet, drives it in 1 inch. Calculate the force of the blow, and the frictional resistance on the side of the pile, per square foot, if all the energy of the blow is thus spent. (I.C.E., Oct., 1906.)

12. State the principle of the Conservation of Energy. In a cable tramway, one car (weight 14 tons) is on a down gradient of 1 in 50, and another (weight 11 tons) is on an up gradient of 1 in 300. The cable connecting

them weighs 8 tons, and is always equally distributed on the two slopes. Find the acceleration and the speed of the cars, due to gravity only, after running 400 feet from rest, if all frictional resistances are neglected.

(I.C.E., Oct., 1906.)

13. Taking the resistances as 13 lbs. per ton, find the horse-power required to produce a speed of 40 miles per hour in a train weighing 300 tons in $3\frac{1}{2}$ minutes (i.) on the level, (ii.) down an incline of 1 in 320.

(I.C.E., Oct., 1906.)

14. A steam hammer weighs 10 tons, and the steam pressure is 50 lbs. per square inch on the piston of 21 inches diameter. Find (i.) the acceleration at which the hammer comes down, (ii.) the hammer's velocity after descending 3 feet, (iii.) the mean force of the blow if the material being worked is compressed $\frac{1}{2}$ inch.

(I.C.E., Feb., 1907.)

15. Find the horse-power required to haul six 10-ton trucks up a gradient of 1 in 20 at $7\frac{1}{2}$ miles per hour. Calculate also how long an engine of this horse-power, weighing 25 tons and with the six trucks attached, would take to develop this speed on the level, the air and other resistances being 11 lbs. per ton in each case.

(I.C.E., Feb., 1907.)

16. A train, weighing 150 tons and running at 30 miles per hour, has the steam cut off and the brakes applied at a certain point. The brakes would bring it to rest on the level in a distance of 300 yards, but it is on an incline of 1 in 100. At what distance would the train come to rest if running (a) up the incline, (b) down the incline?

(I.C.E., Oct., 1907.)

17. In a colliery winding-plant the weight of the cage and its load is 2.7 tons, and the rope is balanced. The depth of the shaft is 500 yards. The cage ascends with a uniform acceleration of 4.5 feet per second per second for 9 seconds; it then ascends at uniform speed, and at the top the retardation is also 4.5 feet per second per second for 9 seconds. Find the time taken to make a journey, and the tensions in the rope during acceleration and retardation.

(I.C.E., Oct., 1907.)

18. What is meant by the term "conservation of energy"? Illustrate your answer by examples. A body weighing 10 lbs. is projected vertically upwards with a velocity of 100 feet per second. Determine its potential and kinetic energy after a lapse of 3 seconds.

(I.C.E., Oct., 1907.)

19. A truck of 10 tons weight, moving horizontally at 4 miles per hour, is brought to rest by the compression of two similar springs. The force due to each spring is, at the beginning, 600 lbs. and increases 300 lbs. for every inch of compression. Find the greatest distance through which each spring is compressed and the maximum force due to each.

(I.C.E., Feb., 1908.)

20. A shot, fired from a gun, has a muzzle velocity of 2,000 feet per second, the direction of flight being at an angle of 30° to the horizontal. Neglecting friction, find the magnitude and direction of its velocity 20 seconds after starting, the highest point in its flight, and its horizontal range.

(I.C.E., Feb., 1908.)

21. An electric tram-car, mass 5 tons, takes 10 H.P. to propel it at 20 miles per hour on the level. What would be the least gradient which the car could descend at the same speed under gravity alone without braking, the friction being assumed the same as on the level? What power would be required to take the car up an incline of 1 in 10 at 20 miles per hour?

(I.C.E., Oct., 1908.)

22. A large hollow sphere, of 20 feet internal radius, is pivoted on a vertical axis and rotated at 20 revolutions per minute. To what position on the inside surface would a ball roll if displaced from the lowest point of the surface? If the coefficient of friction between a wooden block and the

inside surface is 0.2, at what minimum speed must the sphere revolve if the block is to remain at rest in the horizontal diametral plane?

(I.C.E., Oct., 1908.)

23. A swing bridge carried on a pivot weighs 200 tons. The radius of gyration of the mass of the rotating bridge is 25 feet. The diameter of the circular rack by which rotation is effected is 19 feet. The bridge is to be started from rest and uniformly accelerated through an arc of $22\frac{1}{2}^\circ$ in ten seconds. What must be the pressure exerted by the driving-pinion upon the teeth of the circular rack to produce the requisite acceleration? It is found by other calculations that the friction of pivot and gearing and the effect of unbalanced wind require an addition of 100 per cent. to the power as calculated merely for overcoming the inertia. What H.P. is required to operate the bridge at maximum speed attained?

(I.C.E., Oct., 1908.)

LECTURE XII.—I.C.E. QUESTIONS.

1. A flywheel, supported on an axle 2 inches in diameter, is pulled round by a cord wound round the axle and carrying a weight. It is found that a weight of 4 lbs. is just sufficient to overcome friction. A further weight of 16 lbs., making 20 lbs. in all, is applied, and 2 seconds after starting from rest it is found that the weight has descended a distance of 4 feet. Estimate the moment of inertia of the wheel about its axis of rotation, in gravitational units.

(I.C.E., Oct., 1903.)

2. Two masses, of 10 lbs. and 20 lbs. respectively, are attached to a balanced disc at an angular distance apart of 90° and at radii 2 feet and 3 feet respectively. Find the resultant force on the axis when the disc is making 200 turns a minute; and determine the angular position and magnitude of a mass placed at 2.5 feet radius which will make the force on the axis zero at all speeds.

(I.C.E., Oct., 1903.)

3. Estimate the super-elevation which ought to be given to the outer rail when a train moves round a curve of 2,000 feet radius at a speed of 60 miles an hour, the gauge being 4 feet 8½ inches.

(I.C.E., Feb., 1904.)

4. A uniform circular plate, 1 foot in diameter and weighing 4 lbs., is hung in a horizontal plane by three fine parallel cords from the ceiling, and when set into small torsional oscillations about a vertical axis is found to have a period of 3 seconds. A body, whose moment of inertia is required, is laid diametrically across it, and the period is then found to be 5 seconds, the weight of the body being 6 lbs. Find the moment of inertia of the body about the axis of oscillation.

(I.C.E., Feb., 1904.)

5. In a 25-H.P. engine the fluctuation of energy may be taken to be one-quarter the work done in the cylinders per revolution, and the maximum and minimum revolutions of the crank-shaft per minute have to be 131 and 129. If the diameter of the flywheel be 5 feet, estimate the weight of metal which must be concentrated in the rim—the boss and arms of the wheel being neglected.

(I.C.E., Feb., 1904.)

6. A shaft, 10 feet span between the bearings, carries two weights of 10 lbs. and 20 lbs. acting at the extremities of arms 1½ feet and 2 feet long respectively, the planes in which the weights rotate being 4 feet and 8 feet respectively from the left-hand bearing, and the angle between the arms 60° . If the speed of rotation be 100 revolutions per minute, find the displacing forces on the two bearings of the machine. Moreover, if the weights are balanced by two additional rotating weights, each acting at a radius of 1 foot, and placed in planes 1 foot from each bearing respectively,

estimate the magnitudes of the two balance-weights and the angles at which they must be set relative to the two arms. (I.C.E., *Feb.*, 1904.)

7. The faceplate of a lathe has a rectangular slab of cast iron bolted to it, and rotates at 480 revolutions per minute. The slab is 8 inches by 12 inches by 30 inches (the length being radial). Its outside is flush with the edge of the faceplate, which is 48 inches diameter. Find the centrifugal force. (Cast iron weighs $\frac{1}{2}$ lb. per cubic inch.) Where must a circular weight of 300 lbs. be placed to balance the slab? (I.C.E., *Oct.*, 1904.)

8. Find an expression for the radius of gyration of a circular disc. An engine which develops 150 H.P. on the shaft at its normal speed of 75 revolutions per minute (and whose card-area does not alter with the speed) has a disc flywheel 10 feet diameter, weighing 12 tons. How long will it take to get up speed from rest, there being no external load? (I.C.E., *Oct.*, 1904.)

9. In a gas engine of 25 H.P. which runs at 180 revolutions per minute, the greatest fluctuation of energy is equal to the energy exerted in a two-revolution cycle. How many foot-lbs. of energy must be stored in the moving parts of the engine at the mean speed, in order that the fluctuation of speed may not exceed 5 per cent. of the mean? What proportion does this bear to the energy exerted per two-revolution cycle? (I.C.E., *Oct.*, 1904.)

10. A thin circular disc, 12 inches radius, has a projecting axle $\frac{1}{2}$ inch diameter on either side. The ends of this axle rest on two parallel inclined straight edges inclined at a slope of 1 in 40, the lower part of the disc hanging between the two. The disc rolls, from rest, through 1 foot in $53\frac{1}{2}$ seconds. Neglecting the weight of the axle and frictional resistances, find the value of g . (I.C.E., *Feb.*, 1905.)

11. An engine is running at 240 revolutions per minute when the steam is shut off and the load removed at the same instant. The engine runs 300 revolutions before coming to rest. The flywheel weighs 2,000 lbs. and has a radius of gyration of 3 feet. Find the moment of resistance—assumed independent of speed; and if the engine indicate 14 H.P. under normal working conditions, find the mechanical efficiency of the engines—the moment of resistance being assumed independent of load also. (I.C.E., *Feb.*, 1905.)

12. To what stresses are the rims of flywheels exposed:—(i.) When provided with arms, as in the case of a low-speed engine; (ii.) when of the disc type, as in the case of a high-speed engine? Show how these stresses can be calculated in each case. A cast-iron disc flywheel has an outside diameter of 5 feet 6 inches, the width of the rim is 2 feet, and its depth is 1 foot; the flywheel is rotating at 300 revolutions per minute. Calculate the maximum stress and the energy stored in the rim (1 cubic foot of cast iron weighs 470 lbs.). (I.C.E., *Oct.*, 1905.)

13. The flywheel of a gas engine has a radius of gyration of 3 feet, and weighs 1 ton. There is an explosion every 2 revolutions, and the horsepower is 20 at 240 revolutions per minute. Find the energy absorbed by the flywheel during the explosion stroke, and state how you would proceed to determine the percentage fluctuation of speed. (I.C.E., *Oct.*, 1905.)

14. A flywheel weighing 10 tons, whose radius of gyration is 5 feet, rests on bearings 10 inches in diameter. If the coefficient of friction of the axle and bearings is 0.006, find the constant moment which must act upon the wheel to get up a speed of 20 revolutions per minute in one minute. (I.C.E., *Feb.*, 1906.)

15. A flywheel when running at 90 revolutions per minute has a stored energy of 3,000,000 foot-pounds. By reason of additional load it is slowed

down to 86 revolutions per minute in two seconds. By how much will the stored energy be reduced, and what is the average H.P. produced by the slowing down of the flywheel? (I.C.E., Feb., 1906.)

16. The curves of a cycle track have a radius of 120 feet, and are to be banked for a speed of 45 miles per hour. Determine the angle which the track surface must make with the horizontal (i.) when the friction between the wheels and the surface is entirely neglected, (ii.) when the friction is taken as having a minimum coefficient of 0.20.

(I.C.E., Oct., 1906.)

17. The section of a flywheel is as shown in the sketch. Find (i.) its moment of inertia, (ii.) the energy stored in it, and (iii.) the stress in the rim (assumed to be uniform over its area) when turning at 150 revolutions per minute. The weight is 450 lbs. per cubic foot, and the effect of the arms and boss is to be neglected in finding the stress in the rim. (I.C.E., Oct., 1906.)

18. The moment of inertia of a flywheel about the axis of the shaft is 6,000 foot-ton units. The fluctuation of energy per stroke of the piston is 55 foot-tons. Calculate the maximum and the minimum speed of the wheel when its mean speed is 60 revolutions per minute.

(I.C.E., Oct., 1906.)

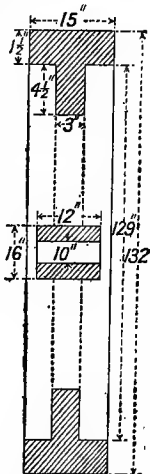
19. An experimental flywheel has an external diameter of 18 inches, the rim is 2 inches wide and 1 inch thick. The arms and boss may be neglected. A weight of 100 lbs. is attached to a cord wound round the axis. Find (i.) the velocity of a point on the outside of the rim if the weight falls 15 feet freely under gravity, neglecting friction, (ii.) how many revolutions the wheel will make before being brought to rest by a braking force of 10 lbs. applied to the rim. Weight of cast iron is 450 lbs. per cubic foot. (I.C.E., Feb., 1907.)

20. A ball, weighing 5 lbs., is connected to a cord 30 inches long, and at the other end is attached to a fixed point. The ball is set rotating round a vertical axis through the point so that its path lies in a horizontal plane, and the radius of the horizontal circle is 10 inches. Determine the speed of rotation and the tension in the cord. (I.C.E., Oct., 1907.)

21. A body rotates round an axis and is subjected to a couple. If there is no frictional resistance, find the relation between the couple acting, the moment of inertia, and the angular acceleration. A flywheel has a moment of inertia of 18, the units being the ton and the foot. Its diameter is 6 feet, and at its circumference a resistance of 1,000 lbs. acts tangentially. In 30 seconds its speed increases from 120 to 150 revolutions per minute. Find the couple acting on the shaft carrying the flywheel. (I.C.E., Oct., 1907.)

22. A rotating drum, 5 feet long and 4 feet in diameter, is out of balance by reason of two weights, one of 10 lbs. at one end of the drum acting at a radius of 2 feet, and the other of 5 lbs. at the other end acting at a radius of 1.5 feet, displaced 90° clockwise as seen in end elevation. Find the position and magnitude of two weights placed on the surface of the drum which will produce correct dynamical balance, one of these weights is to be placed in the same cross-sectional plane as the 10-lb. weight, and the other midway between the two ends of the drum. (I.C.E., Oct., 1907.)

23. Explain why it is necessary to provide reciprocating steam and



gas engines with flywheels. What is meant by the terms coefficient of fluctuation of speed, and coefficient of fluctuation of energy? Obtain a formula embodying these two coefficients, and by its means calculate the weight of rim of a flywheel fulfilling the following conditions:—Mean speed of flywheel rim, 45 feet per second. Indicated H.P., 150. Revolutions per minute, 200. Coefficient of fluctuation of speed, $\frac{1}{50}$. Coefficient of fluctuation of energy, 0.15. (I.C.E., Oct., 1907.)

24. A body, of weight 2 tons, rotates on rails in a vertical circle of radius 30 feet, without friction. Find the least speed at the highest point at which it will retain contact with the rails there, and find then the speed at the lowest point. (I.C.E., Feb., 1908.)

25. A body consists of a circular shaft of 6 inches diameter, having fixed to it concentrically at its centre a circular disc, the radius of gyration of the whole being 18 inches. It rolls, without slipping, with the shaft on a pair of parallel rails inclined at 1 in 20 to the horizontal. Find the distance it travels from rest in 10 seconds, and its kinetic energy then due, (a) to translation, and (b) to rotation. (I.C.E., Feb., 1908.)

26. The flywheel of an engine has a moment of inertia about the axis of the shaft of 8,000 foot-ton units. If the mean speed of the wheel is 50 revolutions per minute, calculate the fluctuation in energy per stroke if the maximum speed is at the rate of 52 and its minimum 48 revolutions per minute. (I.C.E., Oct., 1908.)

LECTURE VIII.—INST.C.E.

2. 1 H.P.

7. 4.6 H.P.

8. 9.4 H.P.

9. $8\frac{7}{8}$ B.H.P.

LECTURE IX.—INST.C.E.

1. 697 lbs.; 60,342 ft.-lbs. 5. 6.3 tons; 332.8 tons.

6. 5.6 lbs.; velocity ratio is 251; mech. advantage is 100.4.

7. M.A. is 412 to 1; T = 20,160 lbs.

8. Pitch is $1\frac{1}{4}$ ins., assuming $\mu = 0.2$.

9. Frictional resistance = 8.3 lbs. per ton; speed on level = 80.86 miles per hour.

11. Horizontal splitting force = 14.3 tons; efficiency = 0.28.

LECTURE X.—INST.C.E.

4. 35.47 feet per second.

6. Velocity of B = 5.8 feet per second. Velocity of point midway = 3.2 feet per second.

7. Force = 17 lbs.; work done = 12.02 ft.-lbs.; velocity = 4.4 feet per sec.

10. Distance = 150 feet.

12. 11.53 radians per second about axis inclined 17.5° to the latter axis.14. (a) 14 miles per hour; (b) 14.5° N. of E.; (c) 7.37 miles per hour; (d) $1\frac{1}{2}^\circ$ towards the W. of N.16. $v_1 : v_2 = 12.9 : 1$.

17. Distance described in 8th second = 240 feet, and in 10th second = 304 feet.

18. R = 13.23 lbs.

19. Maximum velocity 4.24 feet per second; maximum acceleration 53 feet per second per second.

21. Due east at 2.6 miles per hour.

22. (a) 12.3 knots at $13^\circ 18'$ W. of N.; (b) 9.5 knots at 17° N. of W.23. (a) $1 : 1.09$; (b) $\frac{\pi}{2} : 1$.

24. 8 miles.

LECTURE XI.—ORDINARY.

26. K E lost = 19.9 foot-lbs.; $v = 2.35$ feet per second.

27. Average force = 16.45 lbs.; work done = 8.26 foot-lbs.; velocity = 1.63 feet per second.

28. $E_K = \frac{Wv^2}{2g}$. E_K lost in passing through the plate = 21,875 foot-lbs. Distance = 1.4 feet. Average force = 15,625 lbs.

29. K E lost in changing from 100 to 98 r.p.m. = 11,880 foot-lbs. K E at 70 r.p.m. = 255,242 foot-lbs. K E at 71 r.p.m. = 262,048 foot-lbs. K E stored = 7,309 foot-lbs.

30. Work done on the body = 244,538 foot-lbs. $E_K = 129,252$ foot-lbs. v at 10 feet = 30.5 feet per second. v at 30 feet = 71.2 feet per second. Average velocity v (between $x = 45$ and $x = 55$) = 74.4 feet per second; time taken = .135 second.

31. Tractive force = 303 lbs.

32. 366 lbs.; 8 miles per hour.

LECTURE XI.—INST.C.E.

1. 17 lbs.
2. 171 H.P.
3. 8.8 feet per second.
4. 2 minutes 5 seconds. 80 feet per second, or $54\frac{2}{3}$ miles per hour.
5. 9,207 lbs. wt.
7. 25.3 feet per second; 236 foot per second; 22 lbs. wt.
8. 469.75 lbs.; 29,124.8 foot-lbs.; 2.51 H.P.
9. .084 H.P.; .62 H.P.
10. 1,100 lbs.
11. 12.2 lbs.
12. 60.2 feet.
13. 480 lbs.
14. $F = 7.8$ lbs.
15. 2,860 feet.
16. 60.5 feet.
17. $3\frac{1}{8}$ lbs.
18. $v = 2.2$ feet per second; H.P. = .22.
19. $v = 58.8$ feet per second.
20. (i.) 140 lbs.; (ii.) 126 lbs.; (iii.) 105 lbs.
21. $v = 18.36$ feet per second; $F = 61.7$ tons.
23. $1:3.2$.
26. 420.7 H.P.; average resistance = 28.5 tons.
27. Acceleration $\frac{11}{18}$ celo.; slope $\frac{1}{18}$.
28. H.P. = .096; 11.23 lb.-feet.
29. Distance = $80\frac{2}{3}$ feet.
30. $v = 17.9$ feet per second. Average force = 228.6 cwts.
31. 40.42 B.Th.U.
32. Distance = 154.3 feet; velocity = $5\frac{1}{2}$ feet per second; H.P. = .75.
33. Resistance 7.78 tons; ratio 1 : 89.6.

LECTURE XII.—INST.C.E.

1. Slope 1 in 2.58.
2. 30,000 foot-lbs. at mean speed.
3. $I = .45$ unit.
4. 38.2 revolutions per minute.
5. $I = 227,789$ units. $E_K = 1,250$ foot-lbs.
6. $\circ 3.9$ inches, and $\square 1.15$ inches.
7. $I = 0.55$ lb.-ft. unit.
8. $I = \frac{\pi}{4} D^4$; $I = \frac{\pi}{32} A B^2$, where B is length of side and A is the area.
9. $I = \frac{A h^3}{6}$; $I = \frac{5}{12} A B^2$, where B is the side, and A the area.
10. 40 revolutions per minute.
12. 84 and 66 revolutions per minute respectively.
13. 354 revolutions.
14. $E_K = 57.16$ foot-lbs.
15. CF = 8 tons; (a) R = 38.8 tons at $\tan^{-1} \frac{4}{5}$ to the vertical;
- (b) 11.9 inches.
16. 43.1 lbs. per H.P.
17. $v = 15$ miles per hour.
18. W = 36.5 tons

ANSWERS TO QUESTIONS IN APPENDIX B.

LECTURE III.—INST.C.E.

1. 6.6 lbs. weight; 2.73 inches from fulcrum.
2. 26.7 inches from fulcrum.
3. 86.93 lbs. weight; 349.4 lb.-ft.

LECTURE IV.—INST.C.E.

4. 153.85 lbs.
8. 125 lbs.

LECTURE VIII.—INST.C.E.

3. 28.48 B.H.P. ; 207 B.Th.U. per second.

LECTURE IX.—INST.C.E.

3. 14 tons; 0.293.
4. 69.7 H.P.
5. 0.284.
6. 32 lbs. weight; $\mu = 0.106$.

LECTURE X.—INST.C.E.

1. Acceleration = $2\frac{1}{2}$ ft. per second per second ; tractive force = 10.23 tons.
2. Acceleration of train = 2.82 feet per second per second, or 6,922 miles per hour per hour.
3. Velocity of the rain-drops = 21.35 feet per second.
4. Tension in string = $1\frac{1}{3}$ lbs. Distance moved through from rest in 2 seconds by either weight = 21.5 feet.
6. Acceleration = $\frac{60}{741}$ foot per second per second. Velocity at first post = 24.3 feet per second, or 16.56 miles per hour.
7. 14.57 lbs.
8. 3,837 lbs. and 4,647 lbs., assuming the acceleration to continue.
10. At an angle of $46^{\circ} 37'$ to the direction of the train.

LECTURE XI.—INST.C.E.

1. $E_K = 326$ ft.-tons; number of revolutions before coming to rest = 922.
2. Tension in the cable (i.) $843\frac{3}{4}$ lbs.; (ii.) 1,000 lbs.; (iii.) $1,156\frac{1}{2}$ lbs.
3. Total resistance is 174 lbs. per ton; time taken to bring the train to rest is $40\frac{7}{11}$ seconds.
4. Resistance = 8.3 lbs. per ton; and the uniform speed on the level = 81 miles per hour.
5. Acceleration with which the hammer descends = 57 feet per second per second; velocity of hammer = 18.5 feet per second; mean force of compression = 1,852 tons.
6. Kinetic energy stored in the flywheel, $E_K = 97$ foot-lbs. Revolutions of wheel when weight is disconnected = 102; and time it would continue to run against the tangential resistance = 4.8 minutes.
7. 295 ft.; $8\frac{1}{2}$ miles per hour.
8. Acceleration = $\frac{2}{3}g \sin \theta$; $\mu = \frac{1}{3} \tan \theta$.
9. Assuming diameter of axle 6 inches, velocity is $60\frac{1}{2}$ feet per second.
10. 1 ton.
11. 541,800 lbs.; 11,500 lbs. per sq. ft.
12. Acceleration = 0.236 ft. per sec. per sec.; velocity = 13.76 ft. per sec.
13. 520 H.P.; 408 H.P.
14. (i.) 60 ft. per sec. per sec.; (ii.) 19 ft. per sec.; (iii.) 1,362 tons.
15. 148 H.P.; 10.1 seconds.
18. $E_P = 1,560$ ft.-lbs.; $E_K = 2\frac{1}{2}$ ft.-lbs.

LECTURE XII.—INST.C.E.

1. Moment of inertia of flywheel is 1.64 lb.-feet².
2. Resultant force on the axis = 861.7 lbs. Angular position and magnitude of a mass placed at 2.5 feet radius which will make the force on the axis zero at all speeds = 25.3 lbs. at $108^\circ 26'$ to the mass of 10 lbs.
3. Super-elevation given to outer rail = 6.8 inches.
4. Moment of inertia of body about the axis of oscillation = 1.486 lb.-feet².
5. Weight of metal concentrated in rim of flywheel = 2,866 lbs.
6. Displacing forces on the bearings are 43.5 and 117.4 lbs. Magnitude of the two balance weights and the angles at which they must be set relative to the two arms = 12.6 lbs. at $40^\circ 12'$, and $38\frac{1}{2}$ lbs. at $187\frac{1}{2}^\circ$ in advance of the 20-lb. mass and turning in the direction of the 10-lb mass.
7. 43,200 lbs. weight; 21.6 inches radius.
8. $\frac{\tau}{2}$; 3.9 seconds.
10. 32.22 ft. per sec. per sec.
11. 94.25 lb.-ft.; 0.692.
13. 4,125 ft.-lbs.; 1.04 per cent.
14. 667 lb.-ft.
16. $48^\circ 24'$; $37^\circ 7'$.
17. (i.) $I = 108,200$ lb.-ft.², $k = 5.3$ feet; (ii.) $E_K = 414,600$ ft.-lbs.; (iii.) 672 $\frac{1}{2}$ lbs. per sq. inch, taking the velocity of the centre of gyration.
19. 5.6 feet per second; 30.6 revs.

APPENDIX C.

THE INSTITUTION OF CIVIL ENGINEERS' EXAMINATION, FEBRUARY, 1909.

ELECTION OF ASSOCIATE MEMBERS.

APPLIED MECHANICS.

Not more than EIGHT questions to be attempted by any Candidate.

1. When the end of a shaft is struck a blow, a wave of compression travels along the shaft with a definite velocity v , which depends only on the Young's modulus of the material E and on the density of the material ρ . Write down the dimensions of v , E and ρ and from the dimensions obtain the expression for v in terms of E and ρ .

2. Find the horse-power absorbed in friction in a footstep bearing 6-inch diameter carrying a vertical load of 1 ton weight if the speed is 100 revolutions per minute and the coefficient of friction is 0.01. Assume that the pressure is uniformly distributed over the bearing surface.

3. The radius of curvature of a trajectory at a point in the rising branch is 6.85 miles and is inclined at 30° to the vertical. Determine the velocity of the projectile at the point in question. How much higher will the projectile rise? Neglect the air resistance.

4. Show that the natural period of vertical oscillation of a load supported by a spring is the same as the period of a simple pendulum whose length is equal to the static deflection of the spring due to the load.

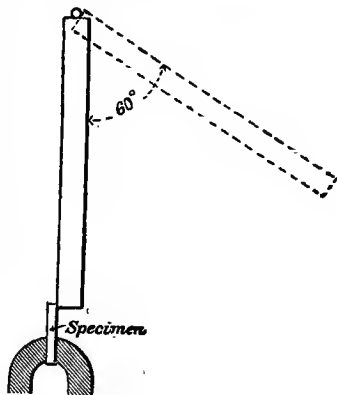
5. Prove that any system of forces acting on a rigid body may be reduced to a single force through any given point and a couple. Also prove that this force and couple can be reduced to a single force.

6. A drawer has a length (a), a depth from back to front (b), and the two handles are symmetrically placed on the front at distance (c) apart. If μ be the coefficient of friction between the drawer and its guides, find the greatest value of c which will allow of the drawer being opened by pulling one handle.

7. A steel shaft 3 inches diameter has two flywheels keyed to it near the ends, the mass of each flywheel being 500 lbs. and the radius of gyration 1 foot. The ends of the shaft rest on two elevated horizontal rails along which the shaft can roll. A rope 1 inch diameter is coiled round the portion of the shaft between the flywheels, one end being fixed to the shaft, and on the free end a load of 500 lbs. is hung. Find the acceleration of the system along the rails.

8. An impact tester for testing steel specimens consists of a compound pendulum made out of a bar 6 feet long of mass 40 lbs., pivoted at the top

end. The bottom end of the bar strikes against the specimen to be tested, which is fixed in a vice immediately below the end of the pendulum in its natural position of rest. The pendulum is deflected through 60° from the vertical, is then set free, and after breaking the specimen it comes to rest at a deflection of 30° on the opposite side of the vertical. Find the velocity of the blow and the energy absorbed by the specimen.



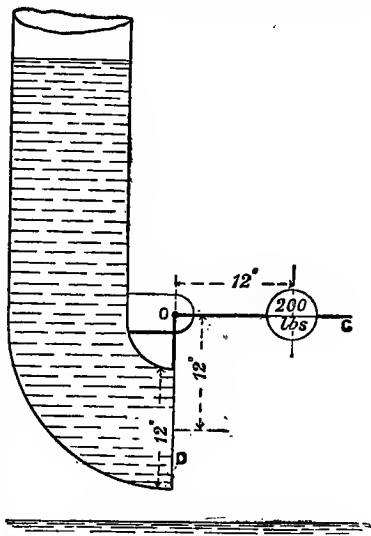
9. Two ships start simultaneously from two ports situated on the same meridian and 50 miles apart. The ship from the more northerly port steers S.W. at 12 knots and the other steers due W. at 15 knots. How near will the two ships approach each other, and at what time after departure will they be closest? (1 knot = 6,080 feet per hour.)

10. Considering a square threaded screw and nut as a case of an inclined plane, show that the efficiency of a simple screw-jack with square threaded screw is given by the following expression, in which α is the slope of the helix and θ the sliding angle ($\tan \theta = \mu$).

$$\eta = \frac{\tan \alpha}{\tan (\alpha + \theta)}.$$

11. Prove that the total pressure on a submerged plane area inclined to the horizontal plane is equal to the product of the area and the unital pressure at the centroid ("centre of gravity") of the area.

12. The outlet from a 12-inch circular pipe is closed as shown in the figure, by a door D hinged at O, which is kept shut by the weight of 200 lbs. acting at 12 inches from the hinge on the bell-crank lever C. Calculate the height of water inside the pipe above the centre of the door, when the door begins to open, the level outside being below the level of the door.



THE INSTITUTION OF CIVIL ENGINEERS'
EXAMINATION, OCTOBER, 1909.

ELECTION OF ASSOCIATE MEMBERS.

APPLIED MECHANICS.

Not more than EIGHT questions to be attempted by any Candidate.

1. What are the relations between the pound, the poundal, and the dyne? Under what circumstances may the pound be used as a unit of force without sensible error? Mention some other cases in which its use would be quite inadmissible.

2. Two weights, A and B, are suspended from the two ends of a light silken cord which passes over a frictionless pulley. By the accelerated descent of the greater weight, A, through a fall of 8 feet, the smaller weight, B, is raised through the same height in 2 seconds of time (starting from a condition of rest). The weight B is 1 lb., how much is the weight A? Neglect the mass of the cord and of the pulley.

3. A horizontal girder 60 feet in length, and carrying a uniform load of 2 tons per foot, is lifted by three hydraulic cylinders which are applied, one under each end and one under the centre of the girder. The three cylinders are all of the same diameter and in hydraulic communication with each other. Find the bending moment at the centre of the girder. Determine also the greatest bending moment and the point where it takes effect upon the girder.

4. A load of 2 tons is suspended by a vertical rope 300 feet long, the rope itself weighing 6 lbs. per foot. In winding up the load to the top, how many foot-lbs. of work are done?

5. A cage, weighing with its load 5 tons, is lifted by a winding engine at the maximum working speed of 30 feet per second. The maximum speed is attained by a uniform acceleration in a period of 6 seconds after starting. Find the tension in the wire rope during this period of time.

6. In the vertical face of a dam, a circular opening 4 feet in diameter is closed by a sluice gate, the water standing 2 feet above the upper edge of the opening. Find the total hydrostatic pressure upon the gate, and the exact height of its centre of pressure.

7. A simple triangular roof truss, ABC, consists of a horizontal tie-beam, BC, 10 feet long, supported at each end, and two inclined rafters, AB and AC, which are respectively 6 feet and 8 feet in length, meeting at the ridge A. Determine the stress in each of the three members due to a load of 1 ton imposed upon the ridge A.

8. A straight horizontal beam, $A B C D$, whose length, $A D$, is 100 feet, and its weight 50 lbs. per lineal foot, is supported and held down to an abutment at A , and supported also at C , 40 feet from A (without being fixed in direction). Find the external forces or reactions at A and C due to the weight of the beam, and also the values of those forces when the beam carries a load of 2,000 lbs. at B , which is 10 feet from A .

9. While a railway train is running at 40 miles an hour upon a falling gradient of 1 in 100 (without steam), the brakes are put on, applying a total retarding force which is equivalent to one-twentieth of the weight of the train. In what distance, and in what space of time, will the train be stopped?

10. A cubical box, 2 feet by 2 feet by 2 feet, partly filled with water, is made to rotate about a vertical axis passing through the centre of its base, at a uniform speed of 60 revolutions per minute. At the centre the water stands 3 inches above the floor of the box. To what height does it rise at each of the four corners? Trace the curve of the water surface upon either of the four vertical walls, writing its equation.

11. A double-armed swing-bridge revolves upon a horizontal turn-table at the centre of its length, being actuated by a chain-drum 32 feet in diameter. The two main girders of the bridge, 180 feet in length, are spaced 30 feet apart, transversely (centre to centre), and each girder has the uniform weight of 10 cwts. per foot. Reduce the inertia of these revolving girder masses to the driving point.

12. Write the equations for the two curves which are described by the diametral section of a forced vortex and of a free vortex respectively.

THE INSTITUTION OF CIVIL ENGINEERS'
EXAMINATION, FEBRUARY, 1910.

ELECTION OF ASSOCIATE MEMBERS.

APPLIED MECHANICS.

Not more than EIGHT questions to be attempted by any Candidate.

1. Give your own definition of stable, indifferent, and unstable equilibrium, illustrating each by such examples as you can think of. In the particular case of a submerged incompressible body, whose weight is exactly equal to its displacement (a submarine), describe in all particulars the condition of the body as regards its equilibrium.

2. A cylindrical gasholder of thin steel plate has a diameter of 80 feet and a height of 40 feet. Its weight of 120,000 lbs. is not counterpoised, and its centre of gravity is 24 feet above the lower edge. Find the necessary depth of water seal; and as the holder is gradually raised, find the exact stage in the process when the vessel would pass from a condition of stable to one of unstable equilibrium—if it were not held between guides.

3. A cylindrical vessel 2 feet in diameter, with the water which it contains, revolves about its vertical axis with the uniform angular velocity of 8 radians per second. Find the slope of the water surface at the outer circumference of the vessel, and also the vertical height of that surface above the vertex of the paraboloid: and prove that each particle of water is in equilibrium under the forces which act upon it. If 1 lb. of water is supplied to this vessel at the centre, at the level of the depressed vertex, and thrown off at the outer rim, how many foot-lbs. of work are done?

4. A horizontal channel of V section, whose sides are inclined at 45° , is closed at the end by a vertical partition. The water surface has a width of 4 feet, and consequently a maximum depth of 2 feet. Calculate the total hydrostatic pressure upon the partition and the height of its centre of pressure.

5. At a height of 4 feet above the level floor of a gate-chamber, a jet of water issues horizontally through a small orifice in the gate, and strikes the floor at a distance of 12 feet from the gate. What is the velocity of the issuing jet?

6. The cables of a suspension bridge hang across a span of 600 feet from tower to tower, with a dip of 50 feet, and carry a uniformly distributed

load of 2 tons per foot of the roadway. At the top of each tower the cable is laid over roller bearings and brought down to the abutment as a backstay at the inclination of 2 horizontal to 1 vertical. Find the direct stress in the backstay and the load upon each tower.

7. A cylindrical water pipe with an internal diameter of 12 inches delivers its discharge of 7 cubic feet per second through a converging conical mouth-piece, whose horizontal length AB is 4 feet, while its diameter tapers uniformly from 12 inches at A to 4 inches at B . The water issues at B under atmospheric pressure: find the pressure at A and at each foot in the length AB , assuming that all transformations of energy are effected without any frictional or other losses.

8. A cyclist running at 20 miles an hour, comes to the foot of a hill which rises at the uniform gradient of 1 in 40. How far will the bicycle run up the gradient without pedalling if the rolling and frictional resistances amount to $\frac{1}{80}$ of its loaded weight?

9. For the earth filling at the back of a vertical retaining wall let the angle of repose ϕ be taken as equivalent to a slope of $1\frac{1}{2}$ to 1. Then to find the horizontal pressure in terms of the vertical pressure at any given depth, determine the ratio $\frac{1 - \sin \phi}{1 + \sin \phi}$ by graphic construction.

10. Referring to the movement of an ordinary connecting-rod in a reciprocating engine, what is meant by the instantaneous centre of rotation? Show how you can determine its position.

11. How may the work of acceleration be shown and measured upon a rectilinear crank effort diagram? Illustrate by hand sketch for a single-cylinder engine running at 120 revolutions per minute under a constant load.

12. In a single-cylinder engine, intended to run at 120 revolutions per minute, the work done at each stroke is 2,200 foot-lbs., while the work of acceleration is 400 foot-lbs. Then for a flywheel 4 feet in diameter what must be the weight of the rim if the speed is not to fluctuate beyond the limits of 119 and 121 revolutions? Neglect the mass of the wheel-arms.

THE INSTITUTION OF CIVIL ENGINEERS'
EXAMINATION, OCTOBER, 1910.

ELECTION OF ASSOCIATE MEMBERS.

MECHANICS.

Not more than EIGHT questions to be attempted by any Candidate.

1. Write down in terms of length, time, and mass the dimensions of velocity, density, acceleration, momentum, force, couple, work, intensity of hydrostatic pressure. Distinguish those which are scalars from those which are vectors.

2. A braced cantilever, subject to the two given loads, is shown in the diagram (Fig. 1). Determine the forces in the four bars of the frame, distinguishing pulls from thrusts.

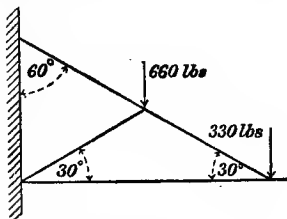


FIG. 1.

3. State the conditions of equilibrium of a system of forces acting in one plane on a rigid body. O A B C is a line drawn on a rigid lamina. At A, B, C forces act, as shown in the diagram (Fig. 2). Find the magnitude, direction, and line of action of a force which will maintain equilibrium with the given forces.

4. In a steam engine, the crank is 18 inches long and the connecting-rod 6 feet long. Consider a position in which the crank and connecting-rod are at right angles, and find the distances of the instantaneous centre of the connecting-rod from the two ends of this rod. If the force on the piston is then 10 tons, find the thrust in the connecting-rod, the force on the slide bars, and the turning moment on the crank-shaft, friction being neglected.

5. The total average force on the pistons of a double-acting marine steam engine of 3 feet 6 inches stroke is 130 tons, the actual thrust delivered by the screw 25 tons, the pitch of the screw 20 feet, and the slip of the screw 10 per cent. Determine the efficiency of the propelling apparatus.

6. A canal of rectangular cross-section, 12 feet broad, is divided into two watertight compartments by a vertical gate coinciding with the cross-section. At one side of the gate there is water 8 feet deep, and at the other side water 3 feet deep. Find the magnitude and line of action of the resultant force on the gate due to the pressure of the water, taking the weight of a cubic foot of water = 62.3 lbs.

7. Ninety cubic feet of water per minute flow through a 6-inch pipe in which there is a right-angled bend; what is the resultant force exerted by the water on the pipe at the bend, neglecting friction?

8. In a differential pulley block the velocity ratio is 30 to 1. When tested, it was found that a pull of 7 lbs. would just raise a load of 24 lbs., and a pull of 25 lbs. a load of 240 lbs. Find the probable pull required to lift 150 lbs. and the efficiency under this load.

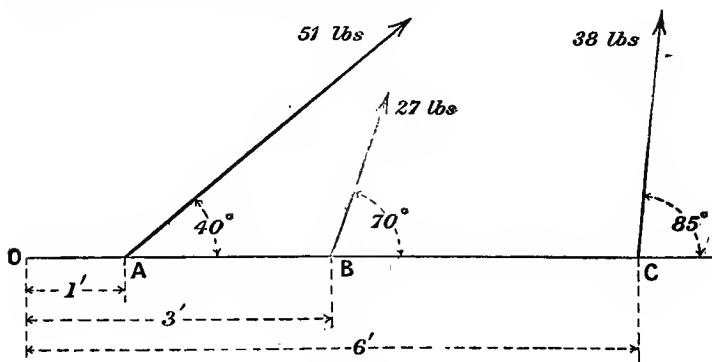


FIG. 2.

9. A projectile is fired with an elevation of 23° and a velocity of 1,560 feet per second. Assuming no air friction, find its range, its time of flight, and the greatest height attained. In what direction, and with what velocity, will the projectile be moving 25 seconds after starting?

10. A load of 5 tons is being hauled by a wire rope up an incline of 1 in 140. Frictional resistance is 60 lbs. per ton. At a certain instant the velocity is 15 miles per hour, and the acceleration up the incline is 1 foot per second per second. Find the pull in the rope and the horse-power exerted at that instant.

11. Show that the moment of inertia of a uniform cylinder, of M mass and radius r , about its axis is $\frac{1}{2} M r^2$. A grindstone, 6 feet in diameter, is making 45 revolutions a minute. If the axle be 2 inches diameter, how long will axle friction take to stop the motion, the coefficient of friction being 0.09?

12. In the case of a rigid body turning about a fixed axis, establish from first principles that angular acceleration = (external couple) \div (moment of inertia about axis). A uniform thin rod of length l is suspended at a point in its length distant $\frac{1}{4} l$ from its centre. It is making small oscillations in a vertical plane. Find the time of a complete oscillation.

1909 MAY EXAMINATION ON SUBJECT VII. APPLIED MECHANICS.

BY THE BOARD OF EDUCATION SECONDARY BRANCH,
SOUTH KENSINGTON, LONDON.

STAGE 2.

INSTRUCTIONS.

Read the General Instructions.

You must not attempt more than *eight* questions in all.

22. Answer only *one* of the following (a), (b), or (c):—(a) See Stage 3, Q. 42, (a). (b) Choose any lifting machine you please; describe how you would measure its efficiency under different loads; what sort of results would you expect to find? (c) A small machine tool is driven direct by an electric motor; how would you determine the horse-power absorbed in the process of cutting the material? (B. of E., S. 2, 1909.)

24. How many foot-lbs. of work must be spent per minute to keep a train, weighing 150 tons, moving at a speed of 25 miles per hour up an incline of 1 in 80, if the frictional resistances are 22 lbs. per ton of load? If the speed of the train while running up this gradient is uniformly increased from 25 miles to 30 miles per hour in 40 seconds, what is the average amount of work done in foot-lbs. per minute during this period of acceleration if the frictional resistance does not alter? (B. of E., S. 2, 1909.)

25. An injection condenser of a steam engine takes its supply of water from a cooling pond, the water level of which is maintained at a height of 18 feet above the condenser inlet. The gauge on the condenser shows a vacuum of 26 inches of mercury. Find, neglecting frictional and other losses, the speed of injection; find also the number of lbs. of water which will be injected per second into the condenser, if the diameter of the injection pipe is $\frac{3}{4}$ inch. The specific gravity of mercury is 13.6. (B. of E., S. 2, 1909.)

26. Four cubic feet of water per minute passing out of a vessel horizontally in a cylindric jet, 1 square inch in section, pass into a second vessel and leave it vertically; what is the momentum per second? What is the horizontal force on each vessel? (B. of E., S. 2, 1909.)

27. In a hinged structure, pieces BO and CO meet at the hinge O, and a force of 2 tons acts upon O in the direction AO. The angle AOB is 115° , BOC is 15° , and the angle AOC is 130° ; find the forces in the two pieces and say whether they are struts or ties. (B. of E., S. 2, 1909.)

28. What is meant by the "bending moment" at a section of a beam? Why is it necessary to know the bending moment? (B. of E., S. 2, 1909.)

29. In finding the total force in the axial direction which a fluid exercises upon a piston or ram, we calculate from the cross-section of the cylinder or ram; why is the actual shape of the face of the piston or end of the ram of no importance? (B. of E., S. 2, 1909.)

30. A body weighing 644 lbs. has the simplest vibrational motion in a straight path, its greatest distance from its middle position being 2 feet. Make a diagram showing what force must act upon it in every position, and state the amounts at the two ends of the path if it makes 150 complete vibrations per minute. (B. of E., S. 2, 1909.)

31. In a single-acting high-speed inverted vertical steam engine, the centre line of the cylinder, when produced, is tangential to the crank circle. The crank revolves with uniform speed at 400 revolutions per minute. Draw a velocity diagram for the piston for one complete revolution of the crank, and from your diagram determine the maximum velocity of the piston on the down and on the up stroke. Length of crank 6 inches, length of connecting-rod 24 inches. (B. of E., S. 2, 1909.)

32. A flat equilateral triangular plate of 4 feet side is supported horizontally by three legs, one at each corner. A vertical force of 112 lbs. is applied to the plate at a point which is distant 3 feet from one leg and 18 inches from another. Determine the compressive force in each leg produced by this load. (B. of E., S. 2, 1909.)

33. Describe, with a sketch, the Venturi water meter, and state the principle of its action. (B. of E., S. 2, 1909.)

STAGE 3.

INSTRUCTIONS.

Read the General Instructions.

You must not attempt more than *eight* questions in all.

41. Describe, with the help of neatly-drawn sketches, which should be roughly to scale, only *one* of the following, (a), (b), (c), or (d):—(a) Any form of loose head-stock or poppet head for a small lathe, with which you have had practical experience; show how the spindle or poppet is advanced or withdrawn, and how it is clamped. (b) A belt gear for giving a slow cutting speed and a quick-return motion, suitable for use in a planing machine, the table of which is traversed to and fro by a screw; show, by assuming dimensions, how you could calculate the speed of cutting and the linear velocity of the table on the return stroke. (c) The making of a wood pattern and the preparation of the mould for a small cast-iron slide-valve of a steam engine. (d) A portable forge; show in detail the means you provide for obtaining the blast. (B. of E., S. 2 and 3, 1909.)

42. Answer only *one* of the following, (a), (b), or (c):—(a) You are provided with a bar of mild steel 40 inches long, of rectangular cross-section 1 inch by $\frac{1}{2}$ inch; explain fully, with sketches of the necessary apparatus, how you would determine by transverse loading of this bar the value of Young's modulus (E) for the material. (b) Describe, with sketches of the apparatus, how you would determine experimentally the coefficient in the formula for friction in a straight length of pipe when water is flowing through it at a steady velocity; you must explain fully the calculations which would be needed; if the pipe was new cast iron, about what value would you expect the coefficient to have? (c) Describe, with sketches of the necessary dynamometer apparatus, how you would measure the cutting forces exerted by lathe tools; you must show in detail how provision is made for free movement of the tool in all three co-ordinate directions, except in so far as it is constrained by the measuring appliances. (B. of E., S. 3, 1909.)

43. An experiment with a small Pelton waterwheel gives results shown in the annexed table :—

Mean Revs. per Minute.	Net Load on Brake Wheel. Lbs.	Water passing through the Turbine in Lbs. per Sec.	Speed of Jet. Feet per Sec. v	Peripheral Speed of Wheel Vanes. Feet per Sec. V	Ratio. $\frac{v}{V}$	Total Kinetic Energy of Jet. Ft.-lbs. per Sec. E	Energy taken up by Brake Wheel. Ft.-lbs. per Sec. e	Efficiency of Wheel per cent. $\frac{e}{E} \times 100$
1,450	0	3·87						
770	12·5	4·20						
660	14·1	4·10						
620	15·1	4·10						
395	19·2	4·12						
121	21·9	4·17						

The cross-sectional area of the nozzle was 0·000764 square foot. The mean diameter of the wheel (centre to centre of buckets) was 10·7 inches. The mean diameter of the brake wheel was 7·25 inches. Fill in the columns left blank in the table. Plot a curve showing the variation of efficiency of wheel with variation of ratio $\frac{v}{V}$. According to simple theory—that is, neglecting all losses—what is the value of $\frac{v}{V}$ which should give the maximum efficiency, and what value would this maximum efficiency be? (B. of E., S. 2 and 3, 1909.)

44. A gun, weighing 50 tons, fires a projectile, weighing $\frac{1}{2}$ ton, with a muzzle velocity of 1,500 feet per second. The diameter of the projectile is 1 foot, and its radius of gyration is 5 inches, the rifling of the gun has a pitch of 60 feet. If the energy of the powder used is 150 foot-tons per lb., find the weight of the powder needed—(a) for the ejection of the projectile, (b) for the rotation of the projectile, (c) for the recoil of the gun. (B. of E., S. 3, 1909.)

45. Give the Euler theory of the strength of a strut. Why is the experimental breaking load always less than what is given by the Euler theory? How do we derive the Gordon or Rankine formula? (B. of E., S. 3, 1909.)

46. Describe the experiments of Osborne Reynolds upon friction of water in a pipe. What was the result of his investigations? In what way does this differ from Darcy's result? (B. of E., S. 3, 1909.)

47. A long beam or arch rib or flying buttress has loads and supporting forces which are known on one side of a cross-section, state clearly how we find the stresses in the section. (B. of E., S. 3, 1909.)

48. A body, weighing 10 lbs., hangs from a spiral spring which elongates 0·01 foot for a pull of 1 lb. Its vibrations are damped, the friction being proportional to the velocity and being 7 lbs. for a speed of 1 foot per second; what is the time of a complete vibration? If the upper end of the

spring is moved up and down with a simple harmonic motion, of amplitude 0.1 foot and with half the natural frequency, what is the nature of the forced vibration after some time has elapsed? (B. of E., S. 3, 1909.)

49. A usual test of a carriage spring is that it may be pressed so that all the strips are straight without taking a set; show that the strips ought all to be made to exactly the same curvature before they are fastened together. (B. of E., S. 3, 1909.)

50. A weight of 420 lbs. is supported by three parallel vertical wires which are attached at their ends to rigid bars; each wire has a diameter of $\frac{1}{8}$ inch; the centre wire is steel and the two outer wires are copper. The three wires are so adjusted that each wire supports a load of 140 lbs. when the air temperature is 40° F. What is the stress per square inch in each wire? If the air temperature rises to 80° F., what is the stress per square inch in each wire? The values of E (Young's modulus) for steel and copper are 30,000,000 and 12,000,000 lbs. per square inch respectively, and the coefficients of expansion per degree Fahrenheit for steel and copper are 0.000006 and 0.000009 respectively. (B. of E., S. 3, 1909.)

51. An inward-flow turbine working under a head of 75 feet has radial blades at the inlet, and the water when it leaves the wheel moves radially. The tip of the blade makes an angle of 30° with the tangent to the periphery at the outlet, and the radial velocity is constant. If the ratio of the radius of the wheel at inlet to the radius at outlet is 1.8 to 1, find the linear velocity at the inner periphery of the wheel. You may neglect all frictional losses. (B. of E., S. 3, 1909.)

52. A hollow steel shaft has to transmit 500 H.P. at a speed of 65 revolutions a minute; the maximum twisting moment in each revolution exceeds the mean by 32 per cent. Determine the necessary dimensions of the cross-section of the shaft, if the shearing stress is not to exceed 5 tons per square inch, and if the internal diameter is to be $\frac{7}{8}$ of the external diameter. (B. of E., S. 3, 1909.)

53. If the tensile stress due to centrifugal force on the rim of a cast-iron pulley is not to exceed 1,200 lbs. per square inch, find the maximum peripheral velocity which is permissible. Cast iron weighs 0.27 lb. per cubic inch. Prove the accuracy of any formula you employ. (B. of E., S. 3, 1909.)

54. Find from the data given below the ratio of the deflection due to shearing and bending stresses respectively in a rolled joist when used as a cantilever, if the load is applied at the extremity of the free end of the cantilever. The joist is 12 inches deep, the flanges are 6 inches wide, the thickness of the web is $\frac{1}{2}$ inch, of the flanges $\frac{3}{4}$ inch. The length of the cantilever is 5 feet, and the load at the free end is 5 tons. $E = 30,000,000$ lbs. per square inch. $N = 12,000,000$ lbs. per square inch. (B. of E., S. 3, 1909.)

1910 MAY EXAMINATION ON SUBJECT VII. APPLIED MECHANICS.*

(b) MACHINES AND HYDRAULICS.

BY THE BOARD OF EDUCATION SECONDARY BRANCH,
SOUTH KENSINGTON, LONDON.

FOR GENERAL INSTRUCTIONS, SEE APPENDIX A.

STAGE 2.†

INSTRUCTIONS.

You must not attempt more than EIGHT questions in all, including Nos. 21 and 22—that is to say, although 21 and 22 are not compulsory, you are not allowed to take more than six questions in addition to Nos. 21 and 22.

7. The pull between locomotive and train is 13 lbs. per ton weight of the train when on the level; the train weighs 200 tons, what is the pull? If the train is being pulled up an incline of 1 in 80 what is now the pull? The speed is 30 miles per hour, what is the horse-power exerted in drawing the train up the incline? (B. of E., S. 2, Div. b, 1910.)

10. A fan drives air vertically downwards through a horizontal circular opening 8 feet in diameter, and so exerts a lifting force of 200 lbs. What is the average downward velocity of the air in the opening? The weight of 1 cubic foot of the air is 0.08 lb. (B. of E., S. 2, Div. b, 1910.)

12. A motor car, when running freely down an incline of 1 in 25, maintains a steady speed of 25 miles per hour. What horse-power would the car engines have to develop to drive the car up the same incline at the same speed? The weight of the car is 3,000 lbs. (B. of E., S. 2, Div. b, 1910.)

15. A flywheel is revolving without friction at 10 radians per second; its kinetic energy is 40,000 ft.-lbs., what is its moment of inertia? A couple of 1,000 pound-feet now acts upon it for a second, what is the increased speed? (B. of E., S. 2, Div. b, 1910.)

STAGE 3.

Read the instructions given under Stage 2.

24. A motor car, whose resistance to motion on the level is supposed to be the same at all speeds, has been running steadily on the level at 20 miles

* Students are referred to Vols. II. and III. for Questions under Div. (a) *Materials and Structures*.

† See Vol. IV.—*Hydraulics*—for Questions 4, 6, 8, 11, 23, 25, and 26, and Vol. V.—*Theory of Machines*—for Questions 3, 5, 9, 13, 14, 21, 22, 28, 30, 33, and 34.

per hour ; it now gets into a rise of 1 in 12 ; what is the maximum length of this rising road which may be traversed by the car without changing gear ? (B. of E., S. 3, Div. b, 1910.)

27. A horizontal lever, instead of having a knife-edge as a fulcrum, is pivoted on a pin 2 inches in diameter. The arms of the lever are 8 inches and 5 feet respectively. The coefficient of friction for the pin is 0.2. What load at the end of the short arm can be raised by a vertical pull of 100 lbs. at the end of the long arm ? (B. of E., S. 3, Div. b, 1910.)

29. What is the limit to the velocity of the rim of an ordinary flywheel ? Does it depend on the diameter ? Prove your statements.

(B. of E., S. 3, Div. b, 1910.)

31. A flywheel revolving without friction at its average speed of 10 radians per second has the kinetic energy 40,000 ft.-lbs., what is its moment of inertia ? The couple $960 \sin 12t$ is acting, what is the speed at any instant ?

(B. of E., S. 3, Div. b, 1910.)

32. Explain clearly why there is loss of energy at a roller bearing. What is known as to the laws of rolling friction ? (B. of E., S. 3, Div. b, 1910.)

APPENDIX D.

THE CENTIMETRE, GRAMME, SECOND, OR C.G.S. SYSTEM OF UNITS OF MEASUREMENT AND THEIR DEFINITIONS.*

I. Fundamental Units.—The C.G.S. and the practical electrical units are derived from the following mechanical units:—

The *Centimetre* as a unit of *length*; the *Gramme* as a unit of *mass*; and the *Second* as a unit of *time*.

The *Centimetre* (cm.) is equal to 0·3937 inch in length, and nominally represents one thousand-millionth part, or $\frac{1}{1,000,000,000}$, of a quadrant of the earth.

The *Gramme* (gm.) is equal to 15·432 grains, and represents the mass of a cubic centimetre of water at 4° C. Also, 1 lb. of 16 ozs. is equal to 453·6 grammes. *Mass* (M) is the quantity of matter in a body.

The *Second* (s) is the time of one swing of a pendulum making 86,164·09 swings in a sidereal day, or the $\frac{1}{86,400}$ part of a mean solar day.

II. Derived Mechanical Units.—

Area (A or cm.²).—The unit of area is the *square centimetre*.

Volume (V or cm.³).—The unit of volume is the *cubic centimetre*.

Velocity (v or cm./s) is rate of change of position. It involves the idea of direction as well as that of magnitude. *Velocity* is *uniform* when equal distances are traversed in equal intervals of time. The unit of velocity is the velocity of a body which moves through unit distance in unit time, or the *velocity of one centimetre per second*.

Momentum (M v, or gm. × cm./s) is the quantity of motion in a body, and is measured by mass × velocity.

Acceleration (a or cm./s²) is the rate of change of velocity, whether that change takes place in the direction of motion or not. The unit of acceleration is the acceleration of a body which undergoes unit change of velocity in unit time, or an acceleration of one centimetre per second per second. The acceleration due to gravity is considerably greater than this, for the velocity imparted by gravity to falling bodies in one second is about 981 centimetres per second (or about 32·2 feet per second). The value differs slightly in different latitudes. At Greenwich the value of the acceleration due to gravity is $g = 981·17$; at the Equator, $g = 978·1$; and at the North Pole, $g = 983·1$.

* The Author is indebted to his Publishers, Charles Griffin & Co., for liberty to abstract the following pages on this subject from the latest edition of Munro and Jamieson's *Pocket-Book of Electrical Rules and Tables for Engineers and Electricians*, to which the student is referred for further values and definitions.—A. J.

Force (*F* or *f*) is that which tends to alter a body's natural state of rest or uniform motion in a straight line.

Force is measured by the rate of change of momentum which it produces, or mass \times acceleration.

The *Unit of Force*, or *Dyne*, is that force which, acting for one second on a mass of one gramme, gives to it a velocity of one centimetre per second. The force with which the earth attracts any mass is usually called the "weight" of that mass, and its value obviously differs at different points of the earth's surface. The force with which a body gravitates—i.e., its weight (in dynes), is found by multiplying its mass (in grammes) by the value of *g* at the particular place where the force is exerted.

Work is the product of a force and the distance through which it acts. The unit of work is the work done in overcoming unit force through unit distance—i.e., in pushing a body through a distance of one centimetre a force of one dyne. It is called the *Erg*. Since the "weight" of 1 gramme is 1×981 or 981 dynes, the work of raising 1 gramme through the height of 1 centimetre against the force of gravity is 981 ergs or *g* ergs. One kilogramme-metre = 100,000 (*g*) ergs. One foot-pound = 13,825 (*g*) ergs = 1.356×10^7 ergs.

Energy is that property which, possessed by a body, gives it the capability of doing work. *Kinetic energy* is the work a body can do in virtue of its motion. *Potential energy* is the work a body can do in virtue of its position. The unit of energy is the *Erg*.

Power or *Activity* is the rate of working. The unit is called the *Watt* (*W*) = 10^7 ergs per second, or the work done at the rate of 1 *Joule* (*J*) per second.

One *Horse-power* (H.P.) = 33,000 ft.-lbs. per minute = 550 ft.-lbs. per second; but, as seen above under *Work*, 1 ft.-lb. = 1.350×10^7 ergs, and, under *Power*, 1 *Watt* = 10^7 ergs per second.

Hence, a *Horse-power* = $550 \times 1.356 \times 10^7$ ergs per sec. = 746 watts.

If *E* = volts, *C* = amperes, and *R* = ohms,

$$\text{then,} \quad \text{H.P.} = \frac{E C}{746} = \frac{C^2 R}{746} = \frac{E^2}{746 R}$$

PRACTICAL ELECTRICAL UNITS.

1. As a **Unit of Resistance (R)**, the **International Ohm (O or ω)**, which is based upon the ohm equal to 10^9 units of resistance of the C.G.S. system of electro-magnetic units, and is represented by the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14.4521 grammes in mass, of a constant cross-sectional area and of the length of 106.3 centimetres.

2. As a **Unit of Current (C or c)**, the **International Ampere (A)**, which is one-tenth of the unit of current of the C.G.S. system of electro-magnetic units, and which is represented sufficiently well for practical use by the unvarying current which, when passed through a solution of nitrate of silver in water, and in accordance with their specifications, deposits silver at the rate of 0.001118 gramme per second.

3. As a **Unit of Electro-motive Force (E)**, the **International Volt (V)**, which is the E.M.F. that, steadily applied to a conductor whose resistance is one International Ohm, will produce a current of one International Ampere, and which is represented sufficiently well for practical use by $\frac{1.083}{1.084}$ of the E.M.F. between the poles or electrodes of the voltaic cell known as Clark's cell, at a temperature of 15° Centigrade, and prepared in the manner described in their specification, or by the *new* Weston cell.

4. As the **Unit of Quantity (Q)**, the **International Coulomb (A \times s)**, which is the quantity of electricity transferred by a current of one International Ampere in one second.

5. As the **Unit of Capacity (K)**, the **International Farad (Fd)** which is the capacity of a conductor charged to a potential of one International Volt by one International Coulomb of electricity.

6. As a **Unit of Work** the **Joule (J)**, or **Watt-second ($W_p \times s$)**, which is 10^7 units of work in the C.G.S. system, and which is represented sufficiently well for practical use by the energy expended in one second in heating an International Ohm.

7. As the **Unit of Power (P_w)**, the **International Watt (W_p)**, which is equal to 10^7 units of power in the C.G.S. system, and which is represented sufficiently well for practical use by the work done at the rate of one Joule per second. The **Kilowatt (Kw.)** = 1,000 Watts = $1\frac{1}{2}$ Horse-power.

8. As the **Unit of Induction (L)**, the **Henry (H)**, which is the induction in the circuit when the E.M.F. induced in this circuit is one International Volt while the inducing current varies at the rate of one ampere per second.

9. The **Board of Trade Commercial Unit of Work (B.T.U.)** is the **Kilowatt-hour (Kw.-hr.)** = 1,000 Watt-hours = $1\frac{1}{2}$ H.P. working for one hour. Or, say, 10 amperes flowing in a circuit for 1 hour at a pressure of 100 volts.

NOTE.—For further simple explanations, with examples, see the latest edition of Prof. Jamieson's *Manual of Practical Magnetism and Electricity*. Also, see the latest edition of Munro and Jamieson's *Electrical Engineering Pocket-Book*—both published by Charles Griffin & Co., London.

APPENDIX D.—(Continued.)

EXAMINATION TABLES.

USEFUL CONSTANTS.

1 Inch = 25·4 millimetres.

1 Gallon = ·1605 cubic foot = 10 lbs. of water at 62° F. ∴ 1 lb. = ·01605 cubic foot.

1 Knot = 6080 feet per hour. 1 Naut = 6080 feet.

Weight of 1 lb. in London = 445,000 dynes.

One pound avoirdupois = 7000 grains = 453·6 grammes.

1 Cubic foot of water weighs 62·3 lbs.

1 Cubic foot of air at 0° C. and 1 atmosphere, weighs ·0807 lb.

1 Cubic foot of Hydrogen at 0° C. and 1 atmosphere, weighs ·00557 lb.

1 Foot-pound = $1·3562 \times 10^7$ ergs.

1 Horse-power-hour = 33000 × 60 foot-pounds.

1 Electrical unit = 1000 watt-hours.

Joule's Equivalent to suit Regnault's H, is $\begin{cases} 774 \text{ ft.-lbs.} = 1 \text{ Fah. unit.} \\ 1393 \text{ ft.-lbs.} = 1 \text{ Cent. } \end{cases}$

1 Horse-power = 33000 foot-pounds per minute = 746 watts.

Volts × amperes = watts.

1 Atmosphere = 14·7 lb. per square inch = 2116 lbs. per square foot = 760 m.m. of mercury = 10^8 dynes per sq. cm. nearly.

A Column of water 2·3 feet high corresponds to a pressure of 1 lb. per square inch.

Absolute temp., $t = \theta^\circ \text{ C.} + 273^\circ\text{·7.}$

Regnault's H = $606\text{·5} + \text{·305 } \theta^\circ \text{ C} = 1082 + \text{·305 } \theta^\circ \text{ F.}$

$p u^{1\text{·0613}} = 479$

$\log_{10} p = 6\text{·1007} - \frac{B}{t} - \frac{C}{t^2}$

where $\log_{10} B = 3\text{·1812}$, $\log_{10} C = 5\text{·0871}$,

p is in pounds per square inch, t is absolute temperature Centigrade,
 u is the volume in cubic feet per pound of steam.

$\pi = 3\text{·1416} = \frac{22}{7} = \frac{355}{113} = 10(\sqrt{3} - \sqrt{2}).$

One radian = 57·3 degrees.

To convert common into Napierian logarithms, multiply by 2·3026.

The base of the Napierian logarithm is $e = 2\text{·7183.}$

The value of g at London = 32·182 feet per second per second.

TABLE OF LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 8 12	17 21 25	29 33 37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 8 11	15 19 23	26 30 34
12	0792	0826	0864	0899	0934	0969	1004	1038	1072	1106	3 7 10	14 17 21	24 28 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 6 10	13 16 19	23 26 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1708	1732	3 6 9	12 15 18	21 24 27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 6 8	11 14 17	20 22 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 5 8	11 13 16	18 21 24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2 5 7	10 12 15	17 20 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 5 6	9 12 14	16 19 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 4 7	9 11 13	16 18 20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2 4 6	8 11 13	15 17 19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2 4 6	8 10 12	14 16 18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2 4 6	8 10 12	14 15 17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2 4 6	7 9 11	13 15 17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2 4 5	7 9 11	12 14 16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2 3 5	7 9 10	12 14 15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2 3 5	7 8 10	11 13 15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2 3 5	6 8 9	11 13 14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2 3 5	6 8 9	11 12 14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1 3 4	6 7 9	10 12 13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1 3 4	6 7 9	10 11 13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1 3 4	6 7 8	10 11 12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1 3 4	5 7 8	9 11 12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1 3 4	5 6 8	9 10 12
34	5315	5328	5340	5353	5366	5379	5391	5403	5416	5428	1 3 4	5 6 8	9 10 11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1 2 4	5 6 7	9 10 11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1 2 4	5 6 7	8 10 11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1 2 3	5 6 7	8 9 10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1 2 3	5 6 7	8 9 10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1 2 3	4 5 7	8 9 10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1 2 3	4 5 6	8 9 10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1 2 3	4 5 6	7 8 9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1 2 3	4 5 6	7 8 9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1 2 3	4 5 6	7 8 9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1 2 3	4 5 6	7 8 9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1 2 3	4 5 6	7 8 9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1 2 3	4 5 6	7 7 8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1 2 3	4 5 5	6 7 8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1 2 3	4 4 5	6 7 8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1 2 3	4 4 5	6 7 8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3	3 4 5	6 7 8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1 2 3	3 4 5	6 7 8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 2	3 4 5	6 7 7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 2	3 4 5	6 6 7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 2	3 4 5	6 6 7

TABLE OF LOGARITHMS.—Continued.

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
56	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1 2 2	3 4 5	5 6 7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1 2 2	3 4 5	5 6 7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1 2 2	3 4 5	5 6 7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1 1 2	3 4 4	5 6 7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1 1 2	3 4 4	5 6 7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1 1 2	3 4 4	5 6 6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1 1 2	3 4 4	5 6 6
62	7924	7931	7938	7946	7952	7959	7966	7973	7980	7987	1 1 2	3 3 4	5 6 6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1 1 2	3 3 4	5 5 6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1 1 2	3 3 4	5 5 6
66	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1 1 2	3 3 4	5 5 6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1 1 2	3 3 4	5 5 6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1 1 2	3 3 4	5 5 6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1 1 2	3 3 4	4 5 6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1 1 2	2 3 4	4 5 6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1 1 2	2 3 4	4 5 6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1 1 2	2 3 4	4 5 5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1 1 2	2 3 4	4 5 5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1 1 2	2 3 4	4 5 5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1 1 2	2 3 4	4 5 5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1 1 2	2 3 3	4 5 5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1 1 2	2 3 3	4 5 5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1 1 2	2 3 3	4 4 5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1 1 2	2 3 3	4 4 5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1 1 2	2 3 3	4 4 5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1 1 2	2 3 3	4 4 5
81	9086	9090	9096	9101	9106	9112	9117	9122	9128	9133	1 1 2	2 3 3	4 4 5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1 1 2	2 3 3	4 4 5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1 1 2	2 3 3	4 4 6
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1 1 2	2 3 3	4 4 6
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1 1 2	2 3 3	4 4 5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1 1 2	2 3 3	4 4 5
87	9396	9400	9405	9410	9415	9420	9425	9430	9435	9440	0 1 1	2 2 3	3 4 4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0 1 1	2 2 3	3 4 4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0 1 1	2 2 3	3 4 4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0 1 1	2 2 3	3 4 4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0 1 1	2 2 3	3 4 4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0 1 1	2 2 3	3 4 4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0 1 1	2 2 3	3 4 4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0 1 1	2 2 3	3 4 4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0 1 1	2 2 3	3 4 4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0 1 1	2 2 3	3 4 4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0 1 1	2 2 3	3 4 4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0 1 1	2 2 3	3 4 4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0 1 1	2 2 3	3 3 4

TABLE OF ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0 0 1	1 1 1	2 2 2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0 0 1	1 1 1	2 2 2
·02	1047	1060	1052	1054	1057	1059	1062	1064	1067	1069	0 0 1	1 1 1	2 2 2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0 0 1	1 1 1	2 2 2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0 1 1	1 1 2	2 2 2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0 1 1	1 1 2	2 2 2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0 1 1	1 1 2	2 2 2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0 1 1	1 1 2	2 2 2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0 1 1	1 1 2	2 2 3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0 1 1	1 1 2	2 2 3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0 1 1	1 1 2	2 2 3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0 1 1	1 2 2	2 2 3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0 1 1	1 2 2	2 2 3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0 1 1	1 2 2	2 3 3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0 1 1	1 2 2	2 3 3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0 1 1	1 2 2	2 3 3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0 1 1	1 2 2	2 3 3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0 1 1	1 2 2	2 3 3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0 1 1	1 2 2	2 3 3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0 1 1	1 2 2	3 3 3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0 1 1	1 2 2	3 3 3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0 1 1	2 2 2	3 3 3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0 1 1	2 2 2	3 3 3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0 1 1	2 2 2	3 3 4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0 1 1	2 2 2	3 3 4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0 1 1	2 2 2	3 3 4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0 1 1	2 2 3	3 3 4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0 1 1	2 2 3	3 3 4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0 1 1	2 2 3	3 4 4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0 1 1	2 2 3	3 4 4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0 1 1	2 2 3	3 4 4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0 1 1	2 2 3	3 4 4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0 1 1	2 2 3	3 4 4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0 1 1	2 2 3	3 4 4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1 1 2	2 3 3	4 4 5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1 1 2	2 3 3	4 4 5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1 1 2	2 3 3	4 4 5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1 1 2	2 3 3	4 4 5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1 1 2	2 3 3	4 4 5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1 1 2	2 3 3	4 4 5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1 1 2	2 3 4	4 6 6
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1 1 2	2 3 4	4 6 6
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1 1 2	2 3 4	4 6 6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1 1 2	3 3 4	4 6 6
·44	2764	2761	2767	2773	2780	2786	2793	2799	2805	2812	1 1 2	3 3 4	4 6 6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1 1 2	3 3 4	5 5 6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1 1 2	3 3 4	5 5 6
·47	2961	2968	2965	2972	2979	2985	2992	2999	3006	3013	1 1 2	3 3 4	5 5 6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1 1 2	3 4 4	5 6 6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1 1 2	3 4 4	5 6 6

TABLE OF ANTILOGARITHMS.—Continued.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
75	5623	5636	5649	5662	5675	5688	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	16	17	19
96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

TABLE OF FUNCTIONS OF ANGLES.

Angle.		Chord.	Sine.	Tangent.	Co-tangent.	Cosine.			
Degrees.	Radians.								
0°	0	000	0	0	∞	1	1.414	1.5708	90°
1	.0175	.017	.0175	.0175	57.2900	.9998	1.402	1.5533	89
2	.0349	.035	.0349	.0349	28.6363	.9994	1.389	1.5359	88
3	.0524	.052	.0523	.0524	19.0811	.9986	1.377	1.5184	87
4	.0698	.070	.0698	.0699	14.3007	.9976	1.364	1.6010	86
5	.0873	.087	.0872	.0875	11.4301	.9962	1.351	1.4835	85
6	.1047	.105	.1045	.1051	9.5144	.9945	1.338	1.4661	84
7	.1222	.122	.1219	.1228	8.1443	.9925	1.325	1.4486	83
8	.1396	.140	.1392	.1406	7.1154	.9903	1.312	1.4312	82
9	.1571	.157	.1564	.1584	6.3138	.9877	1.299	1.4137	81
10	.1745	.174	.1736	.1768	5.6713	.9848	1.286	1.3963	80
11	.1929	.192	.1908	.1944	5.1446	.9816	1.272	1.3788	79
12	.2094	.209	.2079	.2126	4.7046	.9781	1.259	1.3614	78
13	.2269	.226	.2250	.2309	4.3315	.9744	1.245	1.3439	77
14	.2443	.244	.2419	.2493	4.0108	.9703	1.231	1.3265	76
15	.2618	.261	.2588	.2679	3.7321	.9659	1.218	1.3090	75
16	.2793	.278	.2756	.2867	3.4874	.9613	1.204	1.2915	74
17	.2967	.296	.2924	.3057	3.2709	.9563	1.190	1.2741	73
18	.3142	.313	.3090	.3249	3.0777	.9511	1.176	1.2566	72
19	.3316	.330	.3256	.3443	2.9042	.9455	1.161	1.2392	71
20	.3491	.347	.3420	.3640	2.7475	.9397	1.147	1.2217	70
21	.3665	.364	.3584	.3839	2.6051	.9336	1.133	1.2043	69
22	.3840	.382	.3746	.4040	2.4751	.9272	1.118	1.1868	68
23	.4014	.399	.3907	.4245	2.3559	.9205	1.104	1.1694	67
24	.4189	.416	.4067	.4452	2.2460	.9135	1.089	1.1519	66
25	.4363	.433	.4226	.4663	2.1445	.9063	1.075	1.1345	65
26	.4538	.450	.4384	.4877	2.0503	.8988	1.060	1.1170	64
27	.4712	.467	.4540	.5095	1.9626	.8910	1.045	1.0996	63
28	.4887	.484	.4695	.5317	1.8807	.8829	1.030	1.0821	62
29	.5061	.501	.4848	.5543	1.8040	.8746	1.015	1.0647	61
30	.5236	.518	.5000	.5774	1.7321	.8660	1.000	1.0472	60
31	.5411	.534	.5150	.6009	1.6643	.8572	.985	1.0297	59
32	.5585	.551	.5299	.6249	1.6003	.8480	.970	1.0123	58
33	.5760	.568	.5446	.6494	1.5399	.8387	.954	.9948	57
34	.5934	.585	.5592	.6745	1.4826	.8290	.939	.9774	56
35	.6109	.601	.5736	.7002	1.4281	.8192	.923	.9599	55
36	.6283	.618	.5878	.7265	1.3764	.8090	.908	.9425	54
37	.6458	.635	.6018	.7536	1.3270	.7983	.892	.9250	53
38	.6632	.651	.6157	.7813	1.2799	.7880	.877	.9076	52
39	.6807	.668	.6293	.8098	1.2349	.7771	.861	.8901	51
40	.6981	.684	.6428	.8391	1.1918	.7660	.845	.8727	50
41	.7156	.700	.6561	.8693	1.1504	.7547	.829	.8552	49
42	.7330	.717	.6691	.9004	1.1106	.7431	.813	.8378	48
43	.7505	.733	.6820	.9325	1.0724	.7314	.797	.8203	47
44	.7679	.749	.6947	.9667	1.0355	.7193	.781	.8029	46
45°	.7854	.765	.7071	1.0000	1.0000	.7071	.765	.7854	45°
			Cosine.	Co-tangent.	Tangent.	Sine.	Chord.	Radians.	Degrees.
								Angle.	

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